Introduction to the Development and Applications of Process Capability Analysis

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Outline

- **Objectives of Process Capability Indices (PCIs)**
- **An Overview of PCIs**
  - C_p, C_{pk}, C_{pm}, C_{pmk} and C_{PU}, C_{PL} (one-sided)
- **The PCIs and Quality Assurance**
  - Process Consistency, Process Centering (Targeting), Process Yield, Process Loss
- **Sampling Distribution of the Estimated PCIs**
- **Capability Testing Based on PCIs**
- **Applications of PCIs**

Background

- As rapid advancement of the manufacturing technology, suppliers require their products be high quality with very low fraction of nonconformities (often measured in **PPM**, parts per million).

- Traditional methods for measuring fraction of nonconformities become inapplicable since any sample of **reasonable size likely contains no defective product items**.

The probability that contains *at least one defective item* of the collected sample for various process yield and the fraction nonconforming in PPM

NC(PPM)	10,000	5,000	1,000	500	100	10	1
n \ p	0.99	0.995	0.999	0.9995	0.9999	0.99999	0.999999
10	0.096	0.049	0.010	0.005	0.001	0.000	0.000
20	0.182	0.095	0.020	0.010	0.002	0.000	0.000
30	0.260	0.140	0.030	0.015	0.003	0.000	0.000
40	0.331	0.182	0.039	0.020	0.004	0.000	0.000
50	0.395	0.222	0.049	0.025	0.005	0.000	0.000
60	0.453	0.260	0.058	0.030	0.006	0.001	0.000
70	0.505	0.296	0.068	0.034	0.007	0.001	0.000
80	0.552	0.330	0.077	0.039	0.008	0.001	0.000
90	0.595	0.363	0.086	0.044	0.009	0.001	0.000
100	0.634	0.394	0.095	0.049	0.010	0.001	0.000
110	0.669	0.424	0.104	0.054	0.011	0.001	0.000
120	0.701	0.452	0.113	0.058	0.012	0.001	0.000
130	0.729	0.479	0.122	0.063	0.013	0.001	0.000
140	0.755	0.504	0.131	0.068	0.014	0.001	0.000
150	0.779	0.529	0.139	0.072	0.015	0.001	0.000
160	0.800	0.552	0.148	0.077	0.016	0.002	0.000
170	0.819	0.573	0.156	0.082	0.017	0.002	0.000
180	0.836	0.594	0.165	0.086	0.018	0.002	0.000
190	0.852	0.614	0.173	0.091	0.019	0.002	0.000
200	0.866	0.633	0.181	0.095	0.020	0.002	0.000
250	0.919	0.714	0.221	0.118	0.025	0.002	0.000
300	0.951	0.778	0.259	0.139	0.030	0.003	0.000

Sample size required for the sample to contain at least one defective item with prob. 0.95

NC	100000	10000	1000	100	10	1	0.1	ppm
p	0.9	0.99	0.999	0.9999	0.99999	0.999999	0.9999999	
n	29	299	2996	29956	299572	2995731	29957322	

Process Capability Indices (PCIs)

- For this reason, recently developed process capability indices (PCIs), including C_p , C_{PU} , C_{PL} , C_{pk} , C_{pm} and C_{pmk} , have received substantial attention in the manufacturing industries, and become the *common language* for process quality between the customer and the supplier, both internally within the organization and externally.

Objectives of PCIs

Tsui (1997) :

- Compare process performance with specifications
- Compare among different processes (unitless measures)
- Provide information about processes:
  - Process Consistency
  - Process Centering (Targeting)
  - Process Yield
  - Process Loss
- Provide direction for quality improvement

The Index C_p

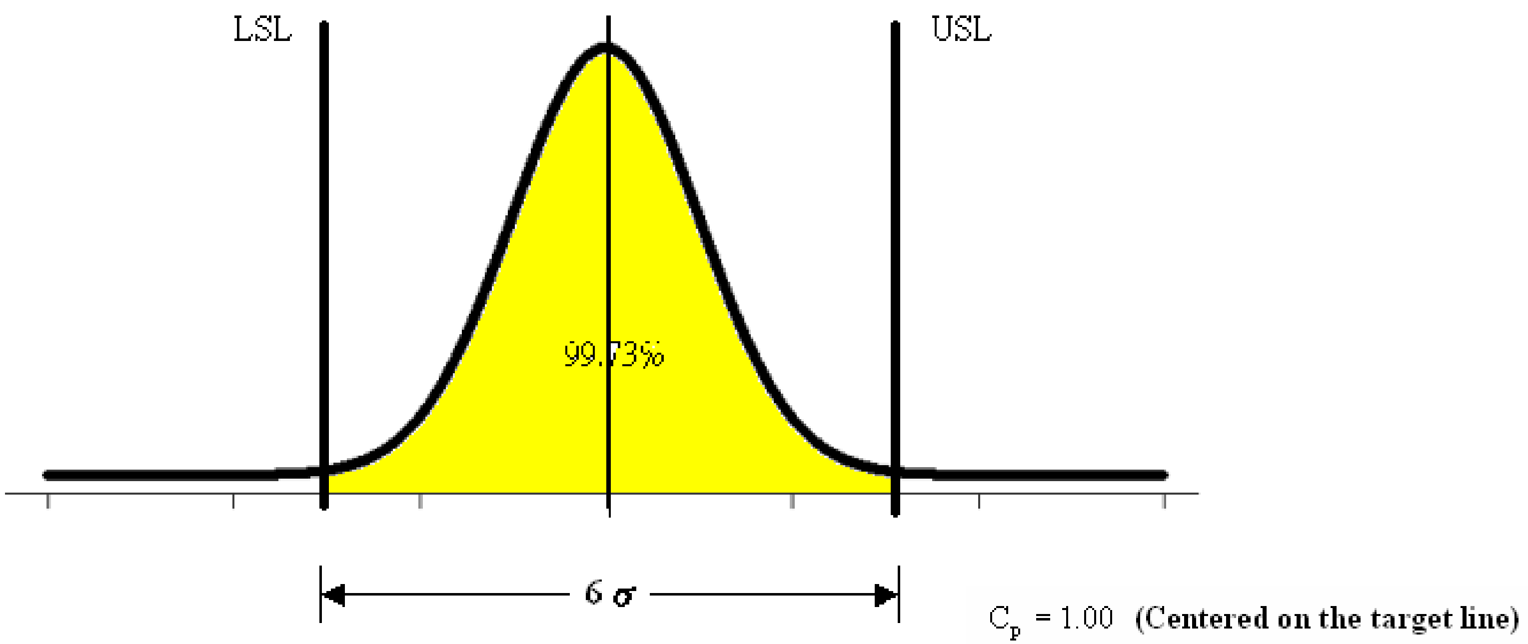
- Definition (Juran (1974), Kane (1986))

$$C_p = \frac{\text{Specification spread}}{\text{Process spread}} = \frac{USL - LSL}{6\sigma},$$

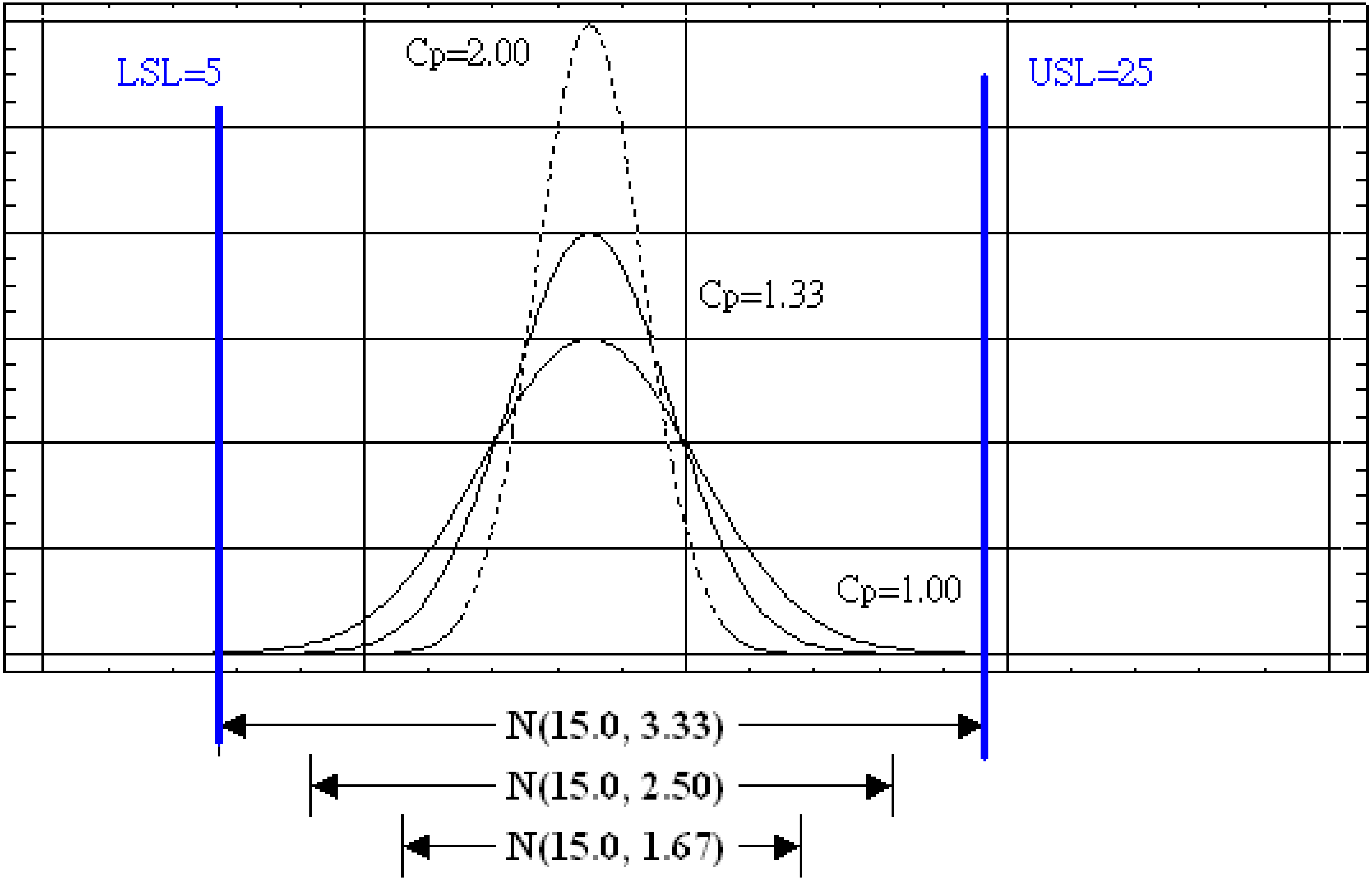
✓Process Capability **Ratio**

✓Process **Precision** Index

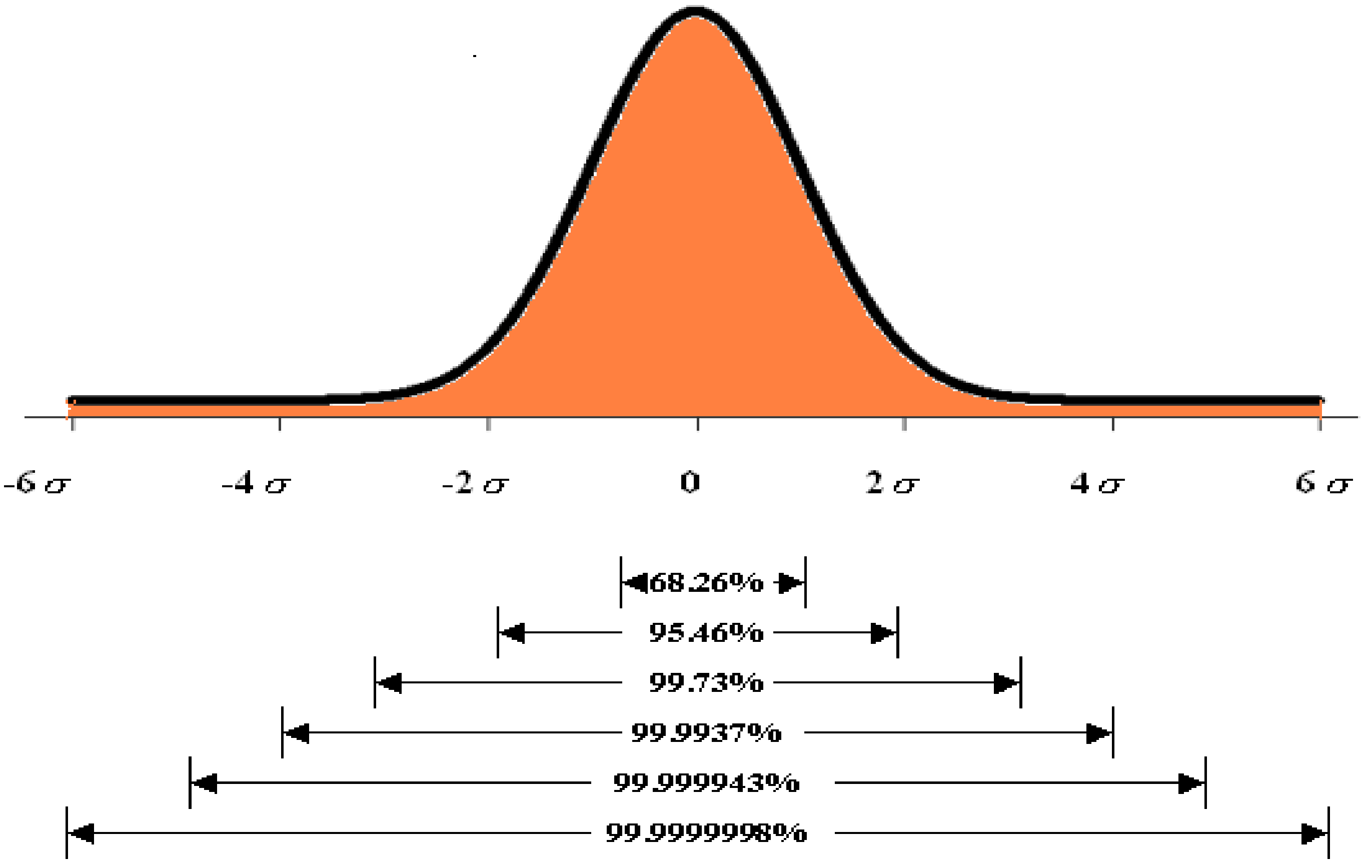
✓Process **Potential** Index



C_p depends on process variation

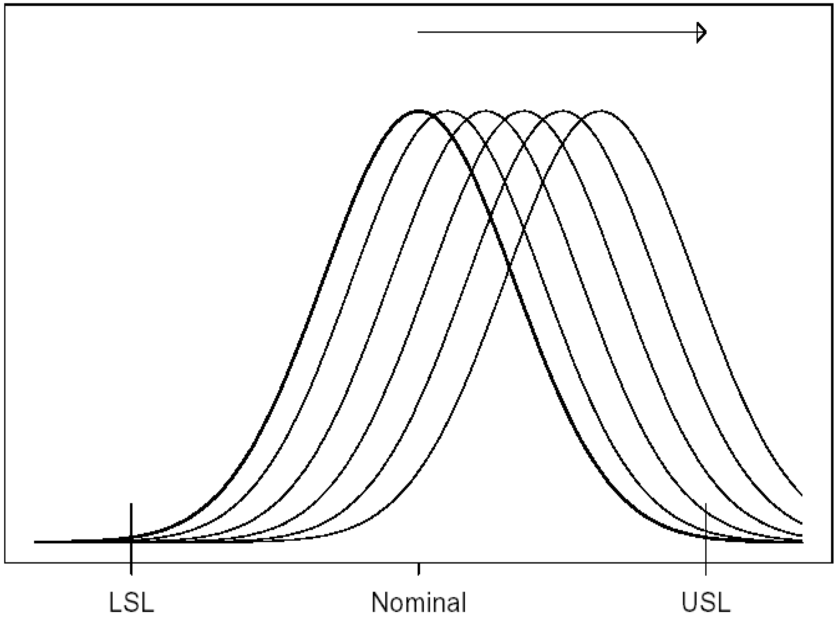


Typical areas under the normal curve



The Weakness of C_p

- The C_p index measures the magnitude of the **overall process variation** relative to the manufacturing tolerance
- It has been criticized for its **inability** to reflect those situations where the process is **not centered** (i.e. $\mu \neq M$)



Various processes with **equal value of C_p**

Capability ratings for various C_p values

$2.00 \le C_p$	Super / Terrific
$1.67 \le C_p < 2.00$	Excellent
$1.33 \le C_p < 1.67$	Good
$1.00 \le C_p < 1.33$	Fair
$0.67 \le C_p < 1.00$	Poor
$0.00 \le C_p < 0.67$	Terrible

The Index C_{pk}

- Definition (Kane (1986))

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}$$

where $d = (USL - LSL)/2, \ M = (USL + LSL)/2$

- The index C_{pk} was developed by taking the magnitude of **process variation** as well as **process location** into consideration.

The Index C_{pk}

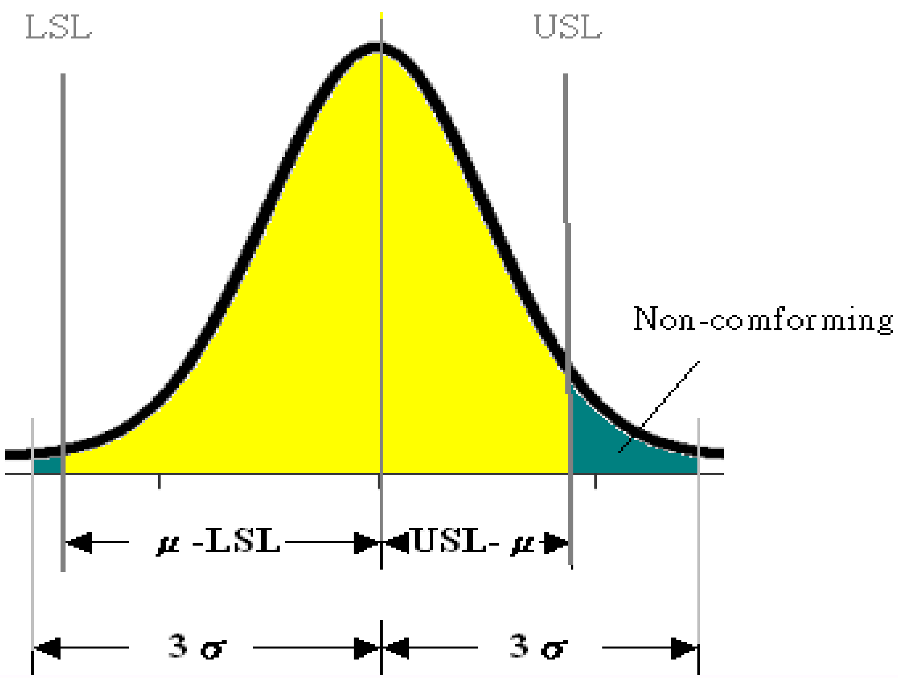
$$C_{pk} = \text{Minimum}(CPU,\ CPL)$$

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\}$$

$$= \frac{d-|\mu-M|}{3\sigma}$$

$$CPU = \frac{\text{Allowable Upper Spread}}{\text{Actual Upper Spread}} = \frac{USL-\mu}{3\sigma}$$

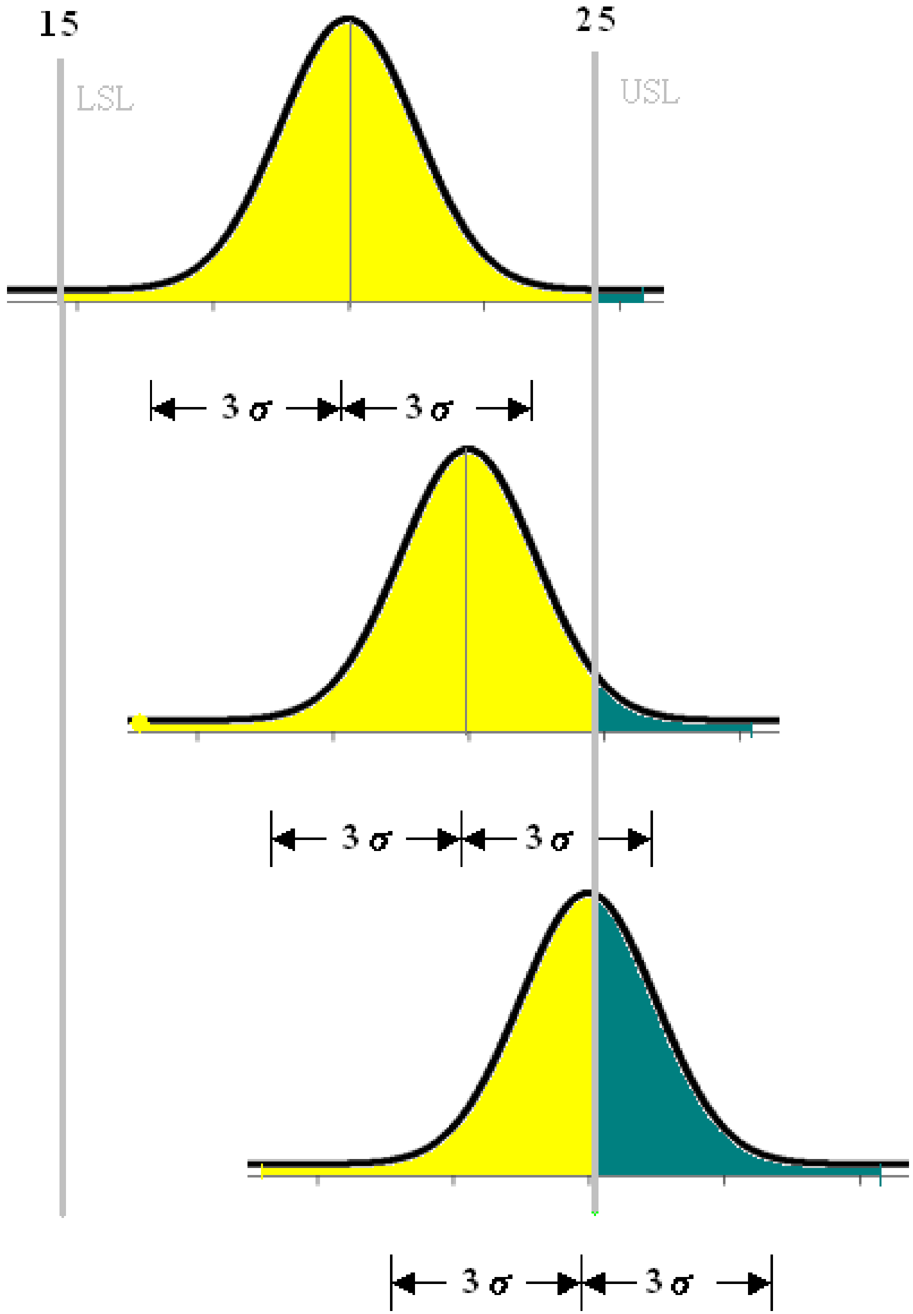
$$CPL = \frac{\text{Allowable Lower Spread}}{\text{Actual Lower Spread}} = \frac{\mu-LSL}{3\sigma}$$



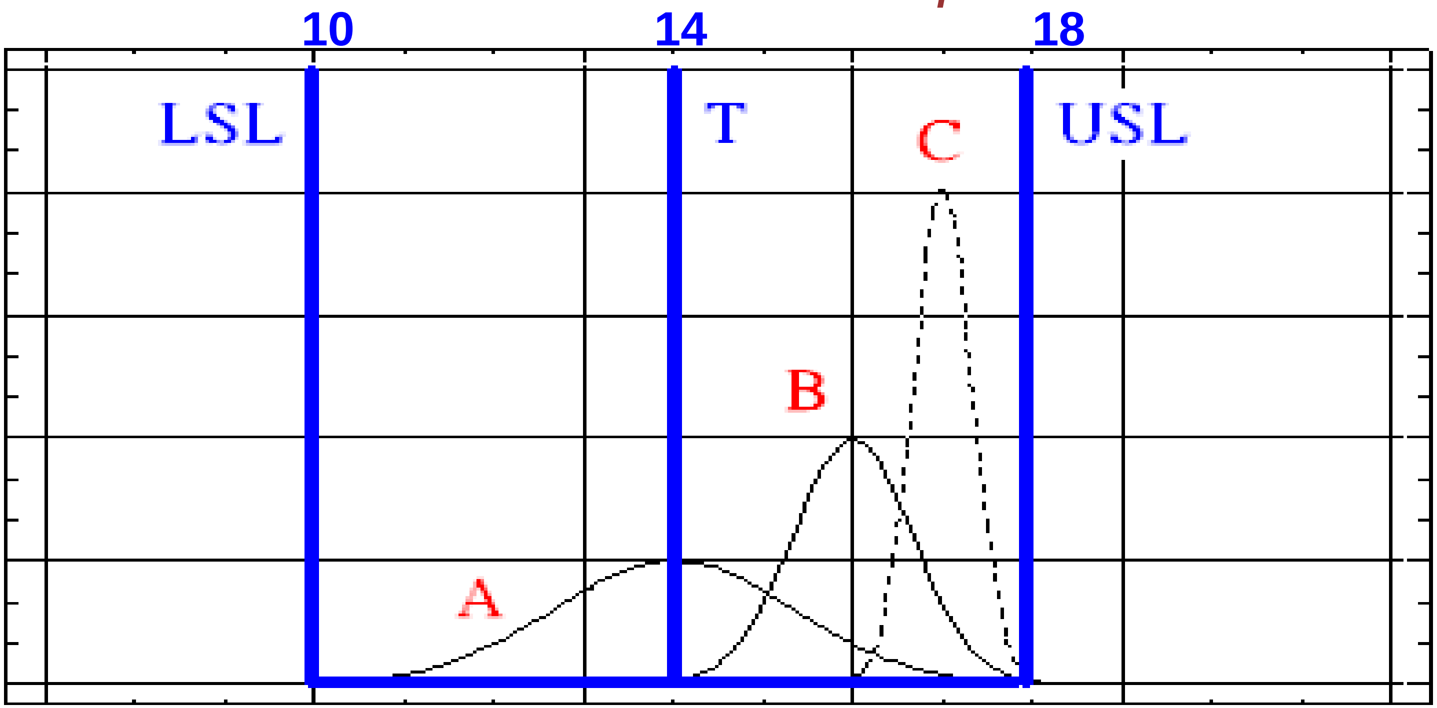
$\mu = 20.0$
$\sigma = 1.25$
$C_p = 1.33$
CPU=1.33
CPL=1.33
$C_{pk} = 1.33$

$\mu = 22.5$
$\sigma = 1.25$
$C_p = 1.33$
CPU=0.67
CPL=2.00
$C_{pk} = 0.67$

$\mu = 25.0$
$\sigma = 1.25$
$C_p = 1.33$
CPU=0.00
CPL=2.67
$C_{pk} = 0.00$



The Index C_{pk}



	μ	σ	C_{pk}
A	14.00	1.33	1.00
B	16.00	0.67	1.00
C	17.00	0.33	1.00

The Index C_{pk}

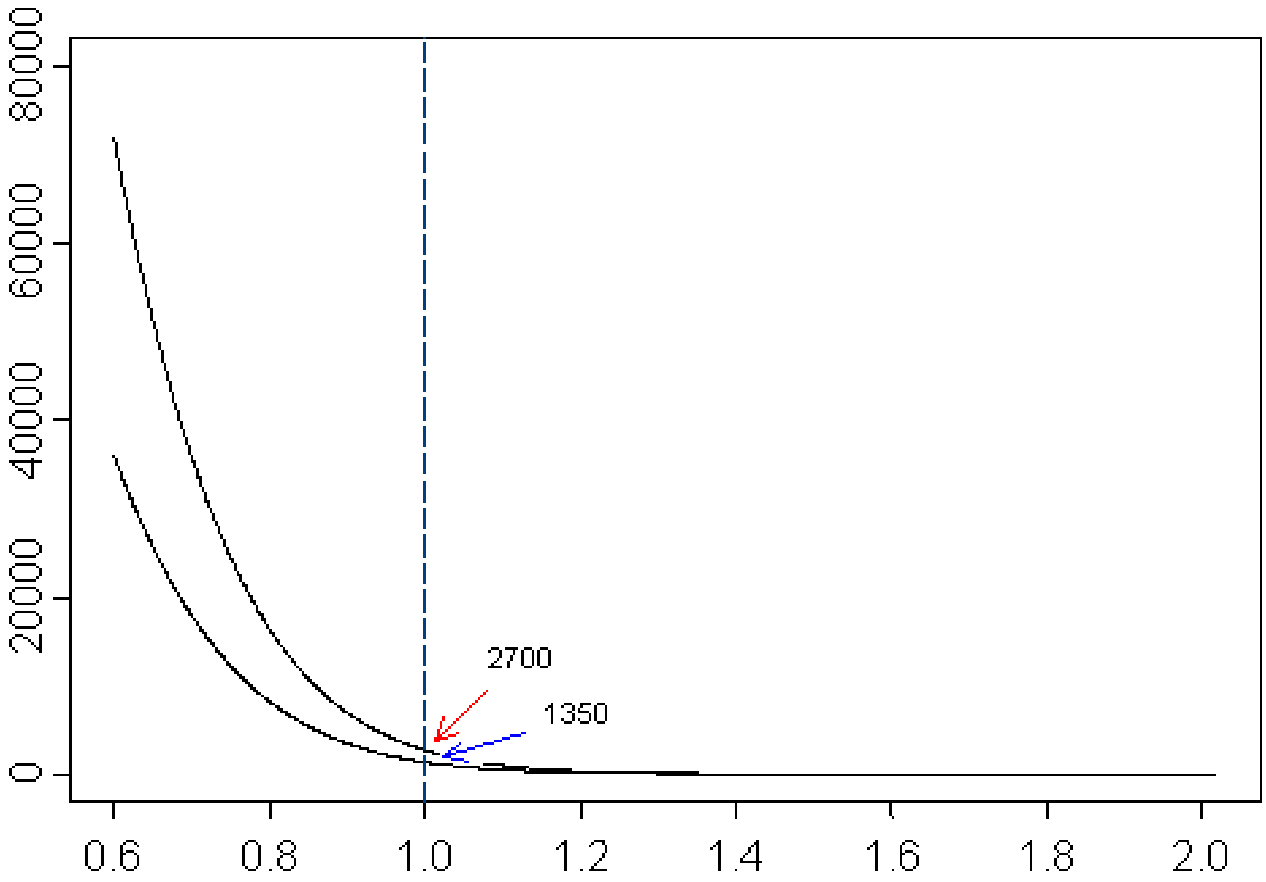
- The index C_{pk} has been regarded as a **yield-based index** since it provides **bounds** on the process yield for a **normally** distributed process given a fixed value of C_{pk}>0 (Boyles(1991)). That is

$$\left[2\Phi(3C_{pk})-1\right]\times 100\% \le \text{yield} \le \Phi(3C_{pk})\times 100\%$$

$$\left[1-\Phi(3C_{pk})\right]\times 10^6 \le \text{NCPPM} \le 2\left[1-\Phi(3C_{pk})\right]\times 10^6$$

The bounds on nonconforming units in PPM versus C_{pk}

Unit: PPM



The bounds on NCPPM versus C_{pk}

The bounds on NCPPM versus C_{pk}

Unit: PPM

C_{pk}	Lower bound	Upper bound
0.60	35930	71861
0.70	17864	35729
0.80	8198	16395
0.90	3467	6934
1.00	1350	2700
1.10	483	967
1.20	159	318
1.24	100	200
1.25	88	177
1.30	48	96
1.33	33	66
1.40	13	27
1.45	6.807	13.614
1.50	3.398	6.795
1.60	0.793	1.587
1.67	0.272	0.544
1.70	0.170	0.340
1.80	0.033	0.067
1.90	0.006	0.012
2.00	0.001	0.002

Some minimum capability requirements of C_{pk} for processes (Montgomery (2005))

C_{pk} Index value	Production process types
1.33	Existing processes
1.50	New processes, or existing processes on safety,strength, or critical parameters
1.67	New processes on safety, strength, or critical parameters

Capability Indices with Sigma Level

σ level	μ = M (best case)		μ Shift 1.5σ (worst case)	
	C_p	NC in PPM	C_{pk}	NC in PPM
3	1	2700	0.5	66810
4	1.33	63	0.833	6210
5	1.67	0.57	1.167	233
6	2	0.002	1.5	3.4

The Indices C_{PU} and C_{PL}

- For the **unilateral** tolerance situation where **only a single** specification limit is given, Kane (1986) considered the following indices as

$$C_{PU} = \frac{USL - \mu}{3\sigma} \qquad C_{PL} = \frac{\mu - LSL}{3\sigma}$$

- The index $\boldsymbol{C_{PU}}$ measures the capability of a **smaller-the-better** process with an upper specification limit ***USL***, whereas the index $\boldsymbol{C_{PL}}$ measures the capability of a **larger-the-better** process with a lower specification limit ***LSL***.

One-sided Indices and Process Yield

- Process **yield** and the indices C_{PU} and C_{PL}

$$P(X < USL) = P\left(\frac{X-\mu}{3\sigma} < \frac{USL-\mu}{3\sigma}\right) = P(Z < 3C_{PU}) = \Phi(3C_{PU})$$

$$P(X > LSL) = P\left(\frac{\mu-X}{3\sigma} < \frac{\mu-LSL}{3\sigma}\right) = P\left(-\frac{1}{3}Z < C_{PL}\right) = \Phi(3C_{PL})$$

- The indices C_{PU} and C_{PL} provide **exact** measures on process yield for processes with **one-sided** specifications

One-sided Indices vs. Process Yield

C_{PU} and the corresponding non-conforming units (in ppm)

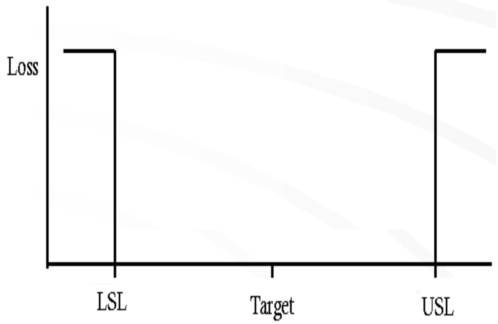
Values of C_{PU}	NCPPM
0.80	8198
1.00	1350
1.25	88
1.45	7
1.60	1
1.80	0.33
2.00	0.00099

Some Minimum Capability Requirements of C_{PU} (or C_{PL}) for Processes

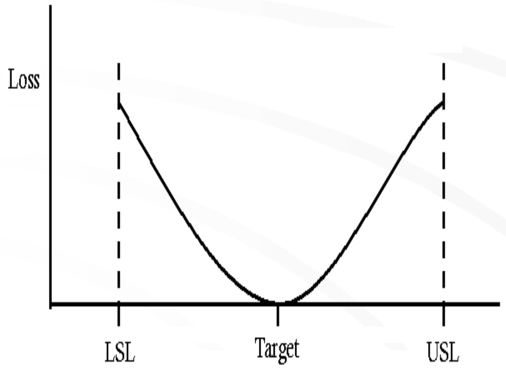
C_{PU} (or C_{PL}) Value	Production Process Type
1.25	Existing Processes
1.45	New Processes; or Existing Processes involving Safety, Strength, or Critical Parameters
1.60	New Processes involving Safety, Strength, or Critical Parameters

The Index C_{pm}

- Definition (Hsiang and Taguchi (1985), Chan *et al.* (1988))

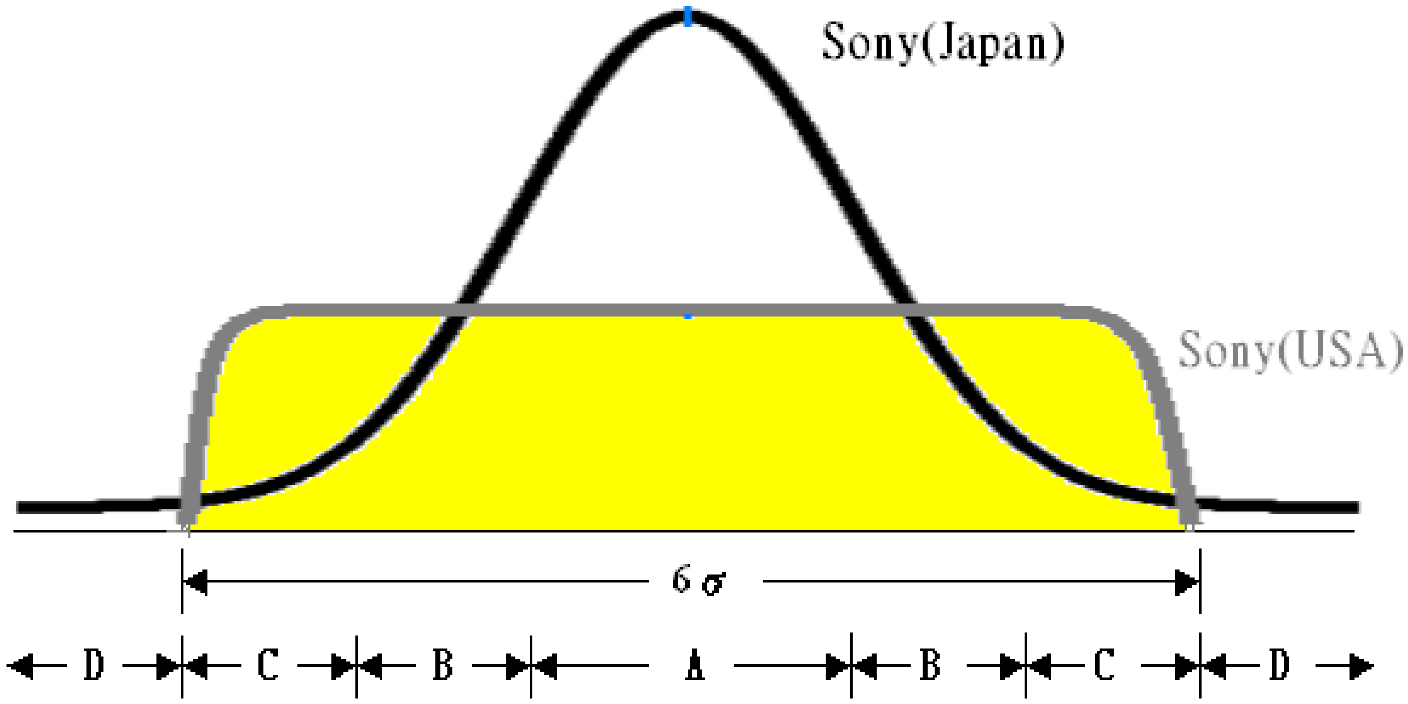


$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{USL - LSL}{6\sqrt{E[(X - T)^2]}}$$



- The item $E[(X-T)^2]$ incorporates **two** variation components:
  - (i) **variation** to the process mean
  - (ii) **deviation** of the process mean from the target
- **Taguchi** capability index, **Loss-based** index

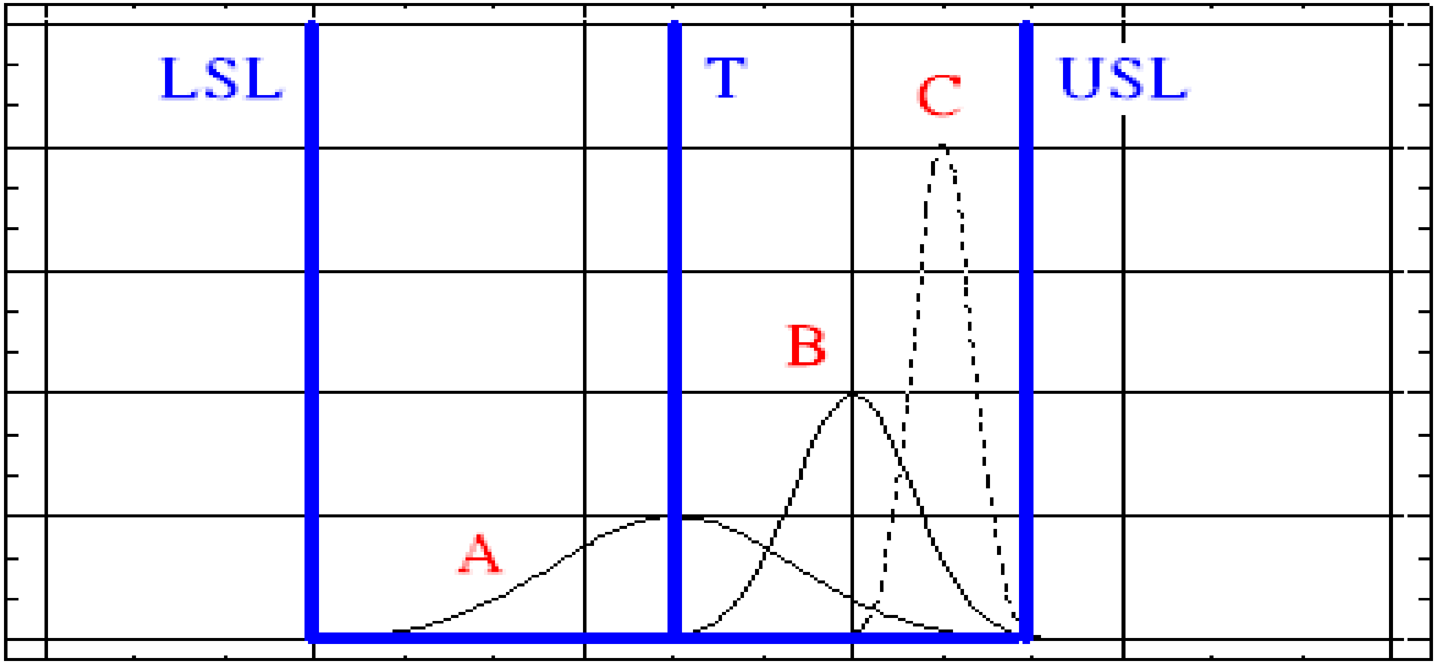
Example: TV sets from Sony
Japan vs. USA



C_{pk} (Japan) < C_{pk} (USA) ----Yield

C_{pm} (Japan) > C_{pm} (USA) ----Loss

The Index C_{pm}



	μ	σ	C_{pk}	C_{pm}
A	14.00	1.33	1.00	1.00
B	16.00	0.67	1.00	0.63
C	17.00	0.33	1.00	0.44

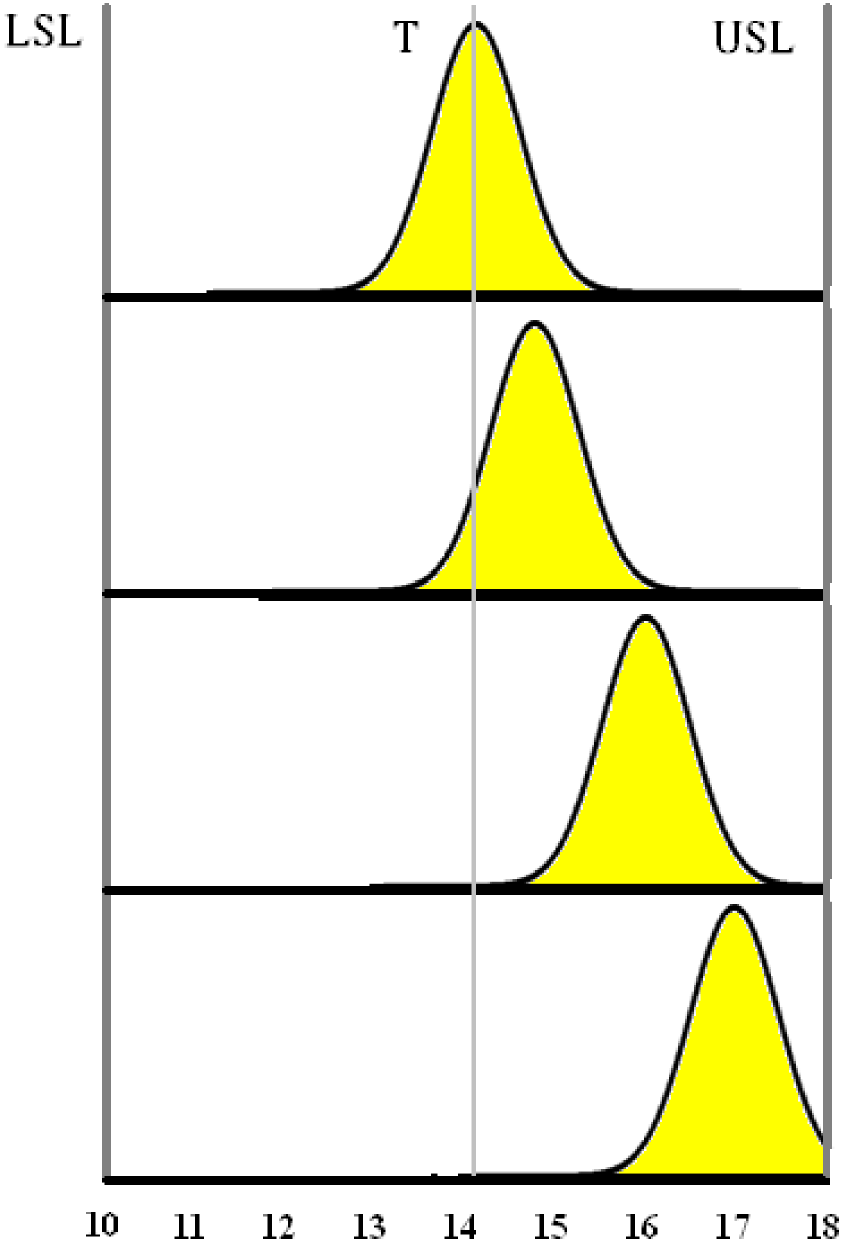
The Index C_{pmk}

- Definition (Pearn *et al.* (1992))

$$C_{pmk} = \min\left\{ \frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}} \right\} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}}$$

$$C_{pmk} = \frac{d-\left|\mu-M\right|}{3\sqrt{\sigma^2+\left(\mu-T\right)^2}} = \frac{C_{pk}\times C_{pm}}{C_p}$$

- The index C_{pmk} alerts the user if the process variance increases and/or the process mean deviates from its target value
- It is constructed by combining the yield-based index $\boldsymbol{C_{pk}}$ and the loss-based index $\boldsymbol{C_{pm}}$
- **Third-generation** capability index



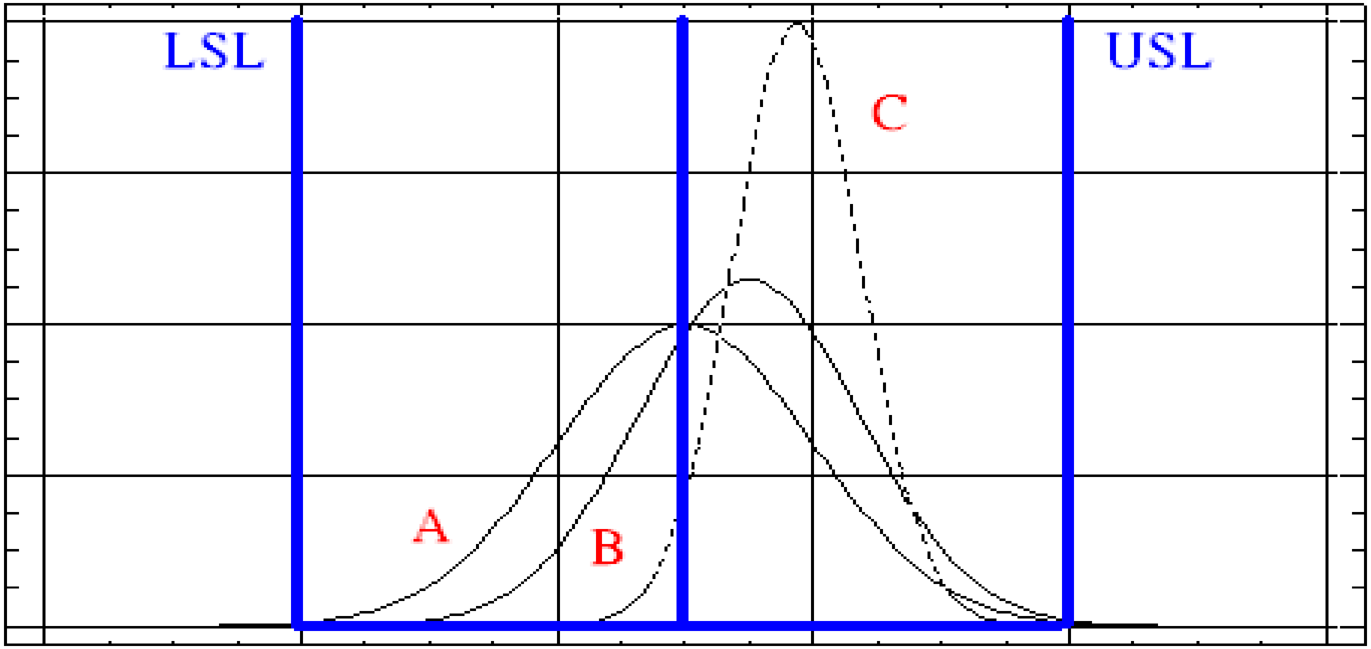
μ	σ	CPL	CPU	C_{pk}	C_{pm}	C_{pmk}
14	2/3	2.0	2.0	2.00	2.00	2.00

μ	σ	CPL	CPU	C_{pk}	C_{pm}	C_{pmk}
15	2/3	2.5	1.5	1.50	1.11	0.83

μ	σ	CPL	CPU	C_{pk}	C_{pm}	C_{pmk}
16	2/3	3.0	1.0	1.00	0.63	0.32

μ	σ	CPL	CPU	C_{pk}	C_{pm}	C_{pmk}
17	2/3	3.5	0.5	0.50	0.43	0.11

C_{pm} vs. C_{pmk}



	μ	σ	C_{pk}	C_{pm}	C_{pmk}
A	13.00	1.00	1.00	1.00	1.00
B	13.50	0.87	0.96	1.00	0.83
C	13.87	0.50	1.42	1.00	0.71

$C_{pk} \geqq C_{pmk}$

$C_{pm} \geqq C_{pmk}$

Relationship among PCIs

$$\begin{tikzcd}[column sep=huge, row sep=huge]
C_p \arrow[r, "\text{Yeild}"] \arrow[d, "\text{Loss}"'] & C_{pk} \arrow[d] \\
C_{pm} \arrow[r] & C_{pmk}
\end{tikzcd}$$

$$C_p = \frac{USL - LSL}{6\sigma} = \frac{d}{3\sigma}$$

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma} = C_p \times \left(1 - \frac{|\mu - M|}{d}\right) = C_p - \frac{1}{3}\left|\frac{\mu - M}{\sigma}\right|$$

$$C_{pm} = \frac{d}{3\sqrt{(\mu - T)^2 + \sigma^2}} = \frac{C_p}{\sqrt{1 + (\frac{\mu - T}{\sigma})^2}}$$

$$C_{pmk} = \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{C_{pk} \times C_{pm}}{C_p}$$

Superstructure of Capability Indices

$$C_p(u,v)=\frac{d-u\left|\mu-M\right|}{3\sqrt{\sigma^2+v(\mu-T)^2}}$$

u \ v	0	1
0	Cp	Cpm
1	Cpk	Cpmk

- Vannman, K., Kotz, S. “A superstructure of Capability Indices: Distribution Properties and Implications,” *Scandinavian Journal of Statistics*, 22, pp. 477-491, 1995.
- Vannman, K., Kotz, S. “A superstructure of Capability Indices: Asymptotics and its Implications,” *International Journal of Reliability, Quality and Satety Engineering*, 2, pp. 343-360, 1995.

The PCIs and Process Yield

$$\%NC \geq 2\Phi(-3C_p) \quad \text{for} \quad C_p \geq 0$$

$$\Phi(-3C_{pk}) \leq \%NC \leq 2\Phi(-3C_{pk}) \quad \text{for} \quad C_{pk} \geq 0$$

$$\%NC \leq 2\Phi(-3C_{pm}) \quad \text{for} \quad C_{pm} \geq \sqrt{3}/3$$

$$\%NC \leq 2\Phi(-3C_{pmk}) \quad \text{for} \quad C_{pmk} \geq \sqrt{2}/3$$

The PCIs and Process Yield

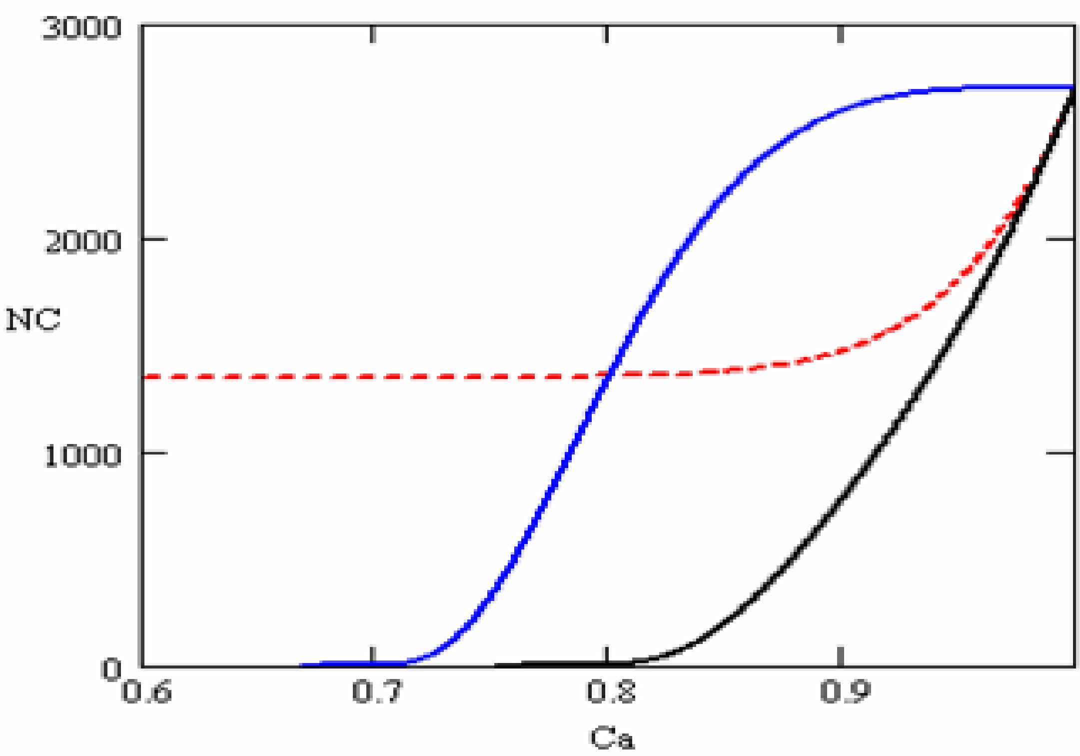


Figure : The actual number of NC curves for $C_{pk} = 1.0$ (top), $C_{pm} = 1.0$ (dash), and $C_{pmk} = 1.0$ (bottom) for various allowed values of C_a.

The PCIs and Process Centering

C_{pk} : $\quad |\mu - T| < d$

C_{pm} : $\quad |\mu - T| < d/(3C_{pm})$

C_{pmk} : $\quad |\mu - T| < d/(3C_{pmk} + 1)$

The PCIs vs. Quality Assurance

	C_p	C_{pk}	C_{pm}	C_{pmk}
σ	V	V	V	V
yield		V	~~V~~	~~V~~
loss		~~V~~	V	V
centering		~~V~~	~~V~~	V

Summary

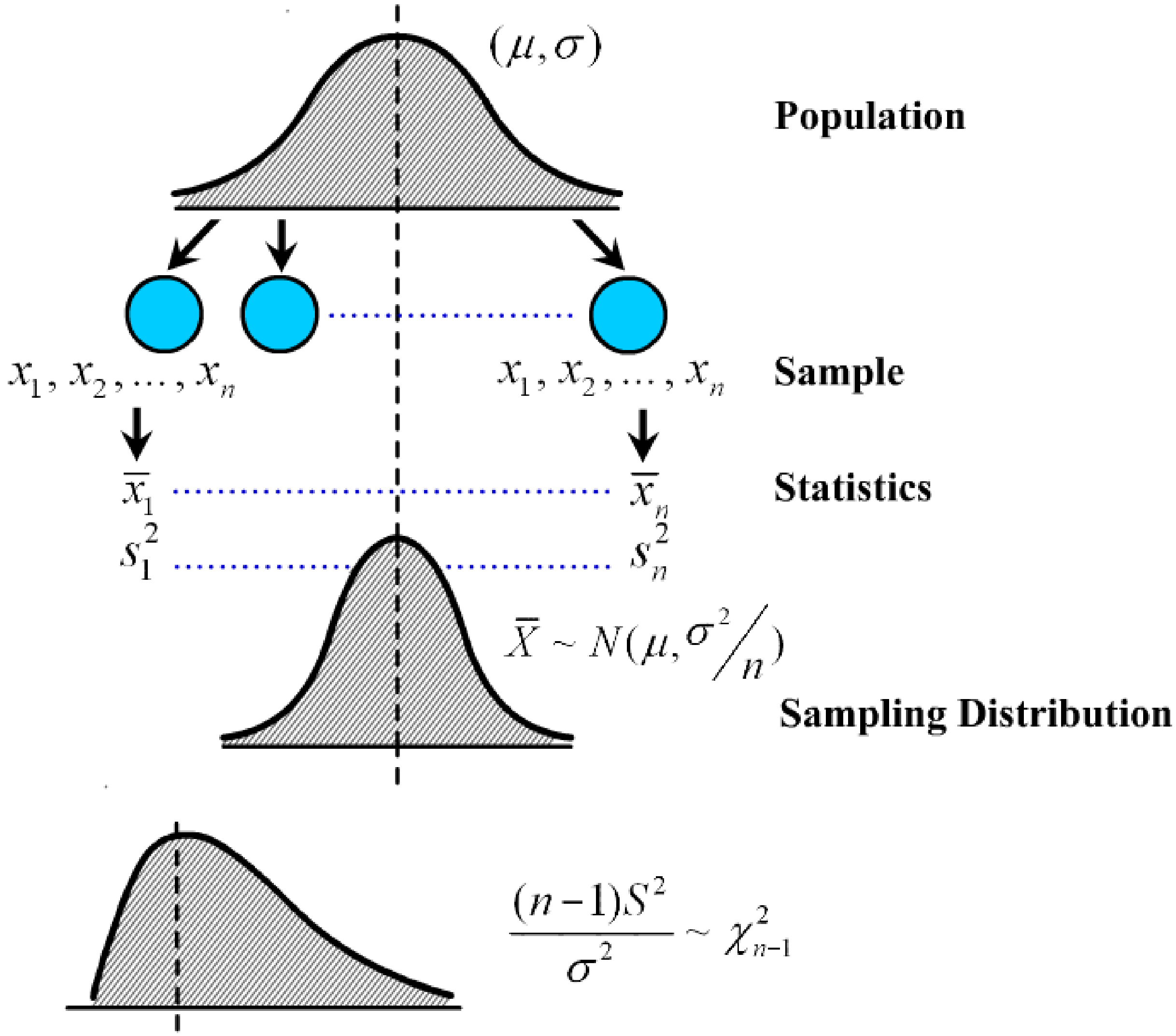
- **The assumptions should be satisfied**
  - **Normality, symmetric tolerances and stable.**
- **$C_p \geqq \max\{C_{pk}、C_{pm}\} \geqq C_{pmk}$.**
  - **That is, $C_p \geqq C_{pk} \geqq C_{pmk}$ and $C_p \geqq C_{pm} \geqq C_{pmk}$.**
- **All indices have their meanings for the quality improvement.**
- **Customers or producers could select a suitable index by their purposes.**

Statistical Inference

- **Statistical Inference**
  - **Estimation**
    - **Point Estimaiton** $\hat{\theta} \to \theta$
    - **Interval Estimation** $\hat{\theta}_L < \theta < \hat{\theta}_U$
  - **Hypothesis Testing**

- **Methods**
  - **Classical Method**
  - **Bayesian**

Statistical Inference



Estimation

- A **point estimate** of those indices is unreliable since the sampling errors have been ignored
- A reliable approach for measuring process capability is to establish **confidence intervals** or via **capability testing**
- In the literature, several authors have promoted the use of various process capability indices and examined with different degrees of completeness. (see Chou *et al.* (1990), Choi and Owen (1990), Bissell (1990), Li *et al.* (1990), Zhang *et al.* (1990), Kirmani *et al.* (1991), Franklin and Wasserman (1992a), Kushler and Hurley (1992), Kotz *et al.* (1993), Kotz and Johnson (1993), Nagata and Nagahata (1994), Chen and Hsu (1995), Vännman and Kotz (1995), Tang *et al.* (1997), Zimmer and Hubele (1997), Vännman (1997), Pearn *et al*. (1998), Hoffman (2001), Zimmer *et al*. (2001), Pearn and Lin (2002, 2003), Pearn and Shu (2003a) for more details).

Estimation of C_p

- Estimation of C_p

$$C_p = \frac{USL - LSL}{6\sigma}$$

$$\hat{C}_p = \frac{USL - LSL}{6s}$$

where $s = [\sum_{i=1}^{n} (x_i - \bar{x})^2 / (n-1)]^{1/2}$ is the sample standard deviation, which can be obtained from a stable process.

- Pearn *et al.* (1998) obtained an unbiased estimator (UMVUE) of C_p by adding a correction factor as

$$\tilde{C}_p = b_{n-1}\hat{C}_p$$

$$b_{n-1} = \sqrt{\tfrac{2}{n-1}}\,\Gamma\left(\tfrac{n-1}{2}\right) / \Gamma\left(\tfrac{n-2}{2}\right)$$

Sampling Distribution of the Estimated C_p

- Under the assumption of normality,

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \Leftrightarrow \frac{\hat{C}_p}{C_p} = \frac{\sigma}{s} \sim \sqrt{\frac{n-1}{\chi^2_{n-1}}}$$

- Chou and Owen (1989) obtained the probability density function (p.d.f.) of

$$f(x) = \frac{2\left[\sqrt{(n-1)/2}\, C_p\right]^{n-1}}{\Gamma\left[(n-1)/2\right]} x^{-n} \exp\left[\frac{-(n-1)\left(C_p\right)^2}{2x^2}\right], \quad x > 0.$$

Confidence Interval for C_p

- Confidence interval for C_p

$$\left[\frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{\sqrt{n-1}}\hat{C}_p,\quad \frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{\sqrt{n-1}}\hat{C}_p\right]$$

$$\text{or}\ \left[\frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{\sqrt{n-1}\,b_{n-1}}\tilde{C}_p,\quad \frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{\sqrt{n-1}\,b_{n-1}}\tilde{C}_p\right],$$

- Critical value for capability testing

H_0: $C_p \le C$ (the process is not capable)

H_1: $C_p > C$ (the process is capable)

$$c_0 = \frac{\sqrt{n-1}\,b_{n-1}}{\sqrt{\chi^2_{n-1,1-\alpha}}}C$$

Estimation of C_p Based on Control Chart Samples

$$\hat{C}_{p(R)} = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}_R}, \quad \hat{\sigma}_R = \frac{\bar{R}_{m,n}}{d_2}.$$

$$\boxed{\bar{R}_{m,n}/\sigma \;\dot{\sim}\; c\chi_v/\sqrt{v}}$$

$$\hat{C}_{p(S)} = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}_S}, \quad \hat{\sigma}_S = \frac{1}{\varepsilon_{n-1}}\bar{S},$$

where $\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$, $S_i = \left[\frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2\right]^{1/2}$

$$\boxed{\hat{\sigma}_S \;\dot{\sim}\; N\left(\sigma, \frac{\sigma^2}{m}\frac{1-\varepsilon_{n-1}^2}{\varepsilon_{n-1}^2}\right)}$$

and $\varepsilon_{n-1} = \sqrt{\frac{2}{n-1}}\frac{\Gamma[n/2]}{\Gamma[(n-1)/2]}$.

Hypothesis testing for C_p based on Xbar-R Chart Samples

$$\alpha = P(\hat{C}_{p(R)} \geq C_{0(R)} \mid C_p = C)$$

$$= P\left(\frac{d}{3\hat{\sigma}_R} \geq C_{0(R)}\right) = P\left(\frac{\bar{R}_{m,n}}{\sigma} \geq \frac{d_2}{C_{0(R)}} \frac{d}{3\sigma}\right)$$

$$\simeq P\left(\chi_v^2 \leq \frac{\sqrt{v}d_2}{c\,C_{0(R)}} C\right).$$

$$C_{0(R)} = \frac{\sqrt{v}\,d_2}{c\sqrt{\chi_{v,\alpha}^2}} C$$

Hypothesis testing for C_p based on Xbar-R Chart Samples

$$1-\alpha = P(\hat{C}_p \geq C_{L(R)}) = P\left(\frac{\hat{\sigma}_R}{\sigma} \geq \frac{C_{L(R)}}{\hat{C}_{p(R)}}\right)$$

$$= P\left(\frac{\bar{R}_{m,n}}{\sigma} \geq \frac{d_2 C_{L(R)}}{\hat{C}_{p(R)}}\right) \simeq P\left(\chi_v^2 \geq \frac{\sqrt{v}\, d_2 C_{L(R)}}{c\,\hat{C}_{p(R)}}\right).$$

$\frac{\sqrt{v}\, d_2 C_{L(R)}}{c\,\hat{C}_{p(R)}} = \chi^2_{v,\alpha}$, or the ratio $\frac{C_{L(R)}}{\hat{C}_{p(R)}} = \frac{c}{\sqrt{v}\, d_2}\sqrt{\chi^2_{v,\alpha}}$

Hypothesis testing for C_p based on Xbar-S Chart Samples

$$\hat{C}_{p(S)} \sim \left[1 + N\left(0, \frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}\right)\right]^{-1} C_p$$

Critical value

$$C_{0(S)} = \frac{C}{1 + z_\alpha \sqrt{\frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}}}$$

Lower confidence limit

$$C_{L(S)} = \hat{C}_{p(S)}\left[1 + z_\alpha \sqrt{\frac{1-\varepsilon_{n-1}^2}{m\varepsilon_{n-1}^2}}\right]$$

Estimation of C_{pk}

- Estimation of C_{pk}

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}$$

$$\hat{C}_{pk} = \min\left\{\frac{USL-\bar{x}}{3s}, \frac{\bar{x}-LSL}{3s}\right\} = \frac{d-|\bar{x}-M|}{3s}$$

$$= \left\{1-\frac{|\bar{x}-M|}{d}\right\}\left(\frac{d}{3s}\right) = \left\{1-\frac{|\bar{x}-M|}{d}\right\}\hat{C}_p$$

$$d = (USL-LSL)/2 \qquad M = (USL+LSL)/2$$

$$\bar{x} = \sum_{i=1}^{n} x_i / n \qquad s = [\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)]^{1/2}$$

Sampling Distribution of the Estimated C_{pk}

The p.d.f. of $\hat{C}_{pk}$ (Chou and Owen (1989))

$$g_{\hat{C}_{pk}}(y) = \begin{cases} 3\sqrt{n}\sum_{i=1}^{2} g_{T_{n-1,\,\delta_i}}(t_i) & \text{if } y \le 0, \\ 3\sqrt{n}\sum_{i=1}^{2} \dfrac{n-1}{t_i}\left[Q_{n+1}\left(\sqrt{\dfrac{n+1}{n-1}}\, t_i, \delta_i; 0, R\right)\right. & \\ \qquad \left. -Q_{n-1}\left(t_i, \delta_i; 0, R\right)\right] & \text{for } y > 0, \end{cases}$$

where $t_1 = -t_2 = 3\sqrt{n}y$, $\delta_1 = -3\sqrt{n}C_{nu}$, $\delta_2 = 3\sqrt{n}C_{nl}$, $R = \sqrt{n-1}(\delta_2 - \delta_1)/(t_2 - t_1)$, and $g_{T_{n-1,\delta_i}}(\cdot)$ is the non-central t distribution with $n-1$ degrees of freedom and the non-centrality parameter δ_i. The function $Q_f(t, \delta; 0, R)$ is given by

$$Q_f(t, \delta; 0, R) = C(f)\int_0^R \left[\Phi\left(\frac{t_x}{f} - \delta\right) x^{f-1}\phi(x) \right] dx \,,$$

and the constant $C(f) = \sqrt{2\pi} / [\Gamma(f/2)2^{f/2-1}]$.

Sampling Distribution of the Estimated C_{pk}

The p.d.f. of $\hat{C}_{pk}$ (Pearn, et. al (1999)):

$$f_{\hat{C}_{pk}}(y) = \begin{cases} 4A_n \sum_{\ell=0}^{\infty} P_\ell(\lambda)\, B_\ell \times \dfrac{D^{n+2\ell}}{a^{2\ell+1}} \int_0^{\infty} (1-yz)^{2\ell} z^{n-1} \\ \quad \times \exp\left\{ -\dfrac{D^2}{18a^2}\left(a^2 z^2 + 9(1-yz)^2\right) \right\} dz, & y \le 0, \\ 4A_n \sum_{\ell=0}^{\infty} P_\ell(\lambda)\, B_\ell \times \dfrac{D^{n+2\ell}}{a^{2\ell+1}} \int_0^{1/y} (1-yz)^{2\ell} z^{n-1} \\ \quad \times \exp\left\{ -\dfrac{D^2}{18a^2}\left(a^2 z^2 + 9(1-yz)^2\right) \right\} dz & y > 0. \end{cases}$$

Here $D = (n-1)^{1/2} d/\sigma$, $a = [(n-1)/n]^{1/2}$, $P_\ell(\lambda) = \dfrac{e^{-(\lambda/2)}(\lambda/2)^\ell}{\ell!}$,

$A_n = \dfrac{1}{3^{n-1}\, 2^{n/2}\, \Gamma((n-1)/2)}$, and $B_\ell = \dfrac{1}{2^\ell\, \Gamma((2\ell+1)/2)}$.

Approximate Confidence Intervals

Heavlin (1988):

$$\left[\hat{C}_{pk} - z_{\alpha/2}\sqrt{\frac{n-1}{9n(n-3)} + \frac{\hat{C}_{pk}^2}{2(n-3)}\left(1+\frac{6}{n-1}\right)},\right.$$

$$\left.\hat{C}_{pk} + z_{\alpha/2}\sqrt{\frac{n-1}{9n(n-3)} + \frac{\hat{C}_{pk}^2}{2(n-3)}\left(1+\frac{6}{n-1}\right)}\right].$$

Zhang *et al.* (1990): $\left[\left(1 - z_{\alpha/2}\sqrt{\frac{n-1}{n-3} - \frac{1}{b_{n-1}^2}}\right)\hat{C}_{pk},\ \left(1 + z_{\alpha/2}\sqrt{\frac{n-1}{n-3} - \frac{1}{b_{n-1}^2}}\right)\hat{C}_{pk}\right]$

Nagata and Nagahata (1992):

$$\left[\hat{C}_{pk} - z_{\alpha/2}\sqrt{\frac{1}{9n} + \frac{\hat{C}_{pk}^2}{2(n-1)}},\ \hat{C}_{pk} + z_{\alpha/2}\sqrt{\frac{1}{9n} + \frac{\hat{C}_{pk}^2}{2(n-1)}}\right].$$

Sampling Distribution of the Estimated C_{pk}

(Pearn et al. (2004))

When $C_{pk} = C$, $b = d/\sigma$ can be expressed as $b = 3C + |\xi|$. Thus, the index C_{pk} can be expressed as a function of the characteristic parameter ξ:

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma} = \frac{d/\sigma - |\xi|}{3},$$

where $\xi = (\mu - M)/\sigma$.

$$F_{\hat{C}_{pk}}(x) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n} - t)^2}{9nx^2}\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right] \mathrm{d}t.$$

for $x > 0$, where $b = d/\sigma$, $\xi = (\mu - m)/\sigma$, $G(\cdot)$ is the c.d.f. of the chi-square distribution with degree of freedom $n-1$, χ^2_{n-1}, and $\phi(\cdot)$ as above is the standard normal p.d.f.

Capability Testing Based on C_{pk}

$$H_0 : C_{pk} \le C \qquad \text{(process is not capable)},$$
$$H_1 : C_{pk} > C \qquad \text{(process is capable)}.$$

$$P(\hat{C}_{pk} > c_0 \mid C_{pk} = C) = \alpha$$

$$\boxed{\int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9nc_0^2}\right)\left[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})\right]dt = \alpha}$$

Critical Value vs Parameter ξ=(μ-M)/σ

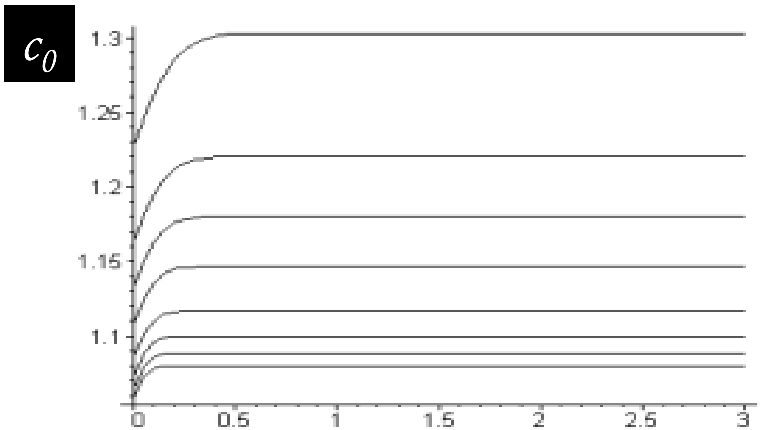


Figure 1. Plots of c_0 versus ξ for $C_{pk} = 1.00$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot).

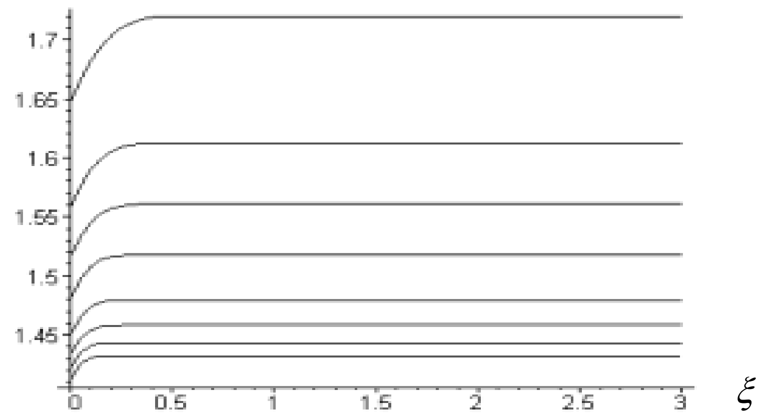


Figure 2. Plots of c_0 versus ξ for $C_{pk} = 1.33$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot).

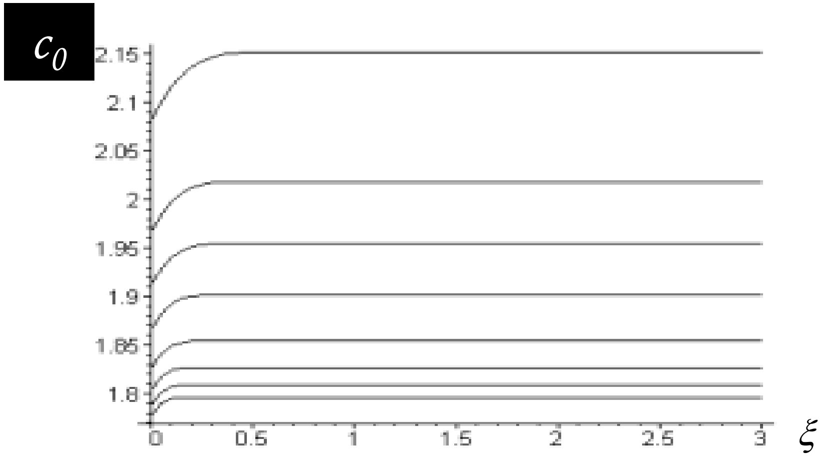


Figure 3. Plots of c_0 versus ξ for $C_{pk} = 1.67$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot).

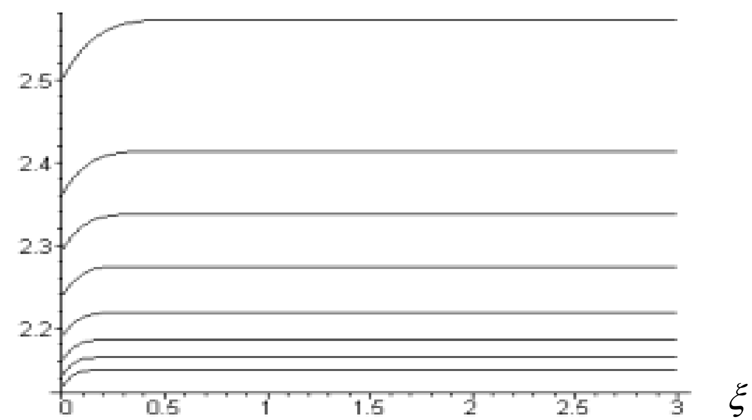


Figure 4. Plots of c_0 versus ξ for $C_{pk} = 2.00$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot).

Lower Confidence Bound on C_{pk}

Hence, given the sample of size n, the confidence level γ, the estimated value $\hat{C}_{pk}$ and the parameter ξ, using numerical integration technique with iterations, the $100\gamma\%$ lower confidence bounds for C_{pk}, C_L, where $b_L = 3C_L + |\xi|$, can be obtained by solving the following equation:

$$\boxed{\int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)(b_L\sqrt{n}-t)^2}{9n\hat{C}_{pk}^2}\right)[\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n})]\,\mathrm{d}t = 1-\gamma.}$$

P-Value

Given $C_{pk} = C$, $b = d/\sigma$ can be expressed as $b = 3C + |\xi|$, the p-value corresponding to c^*, a specific value of $\hat{C}_{pk}$ calculated from the sample data, is:

$$\boxed{\begin{aligned} p-value &= \Pr(\hat{C}_{pk} \geq c^* \mid C_{pk} = C) \\ &= \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9n(c^*)^2}\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right]dt\,. \end{aligned}}$$

Critical Values for C_{pk}=1.00

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	2.141	1.877	1.686
20	1.621	1.469	1.399
30	1.461	1.399	1.303
35	1.415	1.337	1.275
40	1.379	1.309	1.252
45	1.351	1.286	1.235
50	1.328	1.268	1.220
55	1.308	1.253	1.208
60	1.292	1.240	1.197
65	1.278	1.228	1.188
70	1.266	1.219	1.180
75	1.255	1.210	1.173
80	1.245	1.202	1.167
85	1.236	1.195	1.161
90	1.228	1.188	1.156
95	1.221	1.183	1.151
100	1.214	1.177	1.147
110	1.203	1.168	1.139
120	1.193	1.160	1.132

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
130	1.184	1.153	1.126
135	1.180	1.149	1.124
140	1.176	1.146	1.121
145	1.173	1.143	1.119
150	1.170	1.141	1.117
155	1.166	1.138	1.115
160	1.163	1.136	1.113
165	1.161	1.133	1.111
170	1.158	1.131	1.109
175	1.155	1.129	1.107
180	1.153	1.127	1.106
185	1.151	1.125	1.104
190	1.148	1.123	1.103
195	1.146	1.122	1.101
200	1.144	1.120	1.100
210	1.140	1.117	1.097
220	1.137	1.114	1.095

Critical Values for C_{pk}=1.33

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	2.811	2.468	2.220
15	2.345	2.129	1.967
20	2.132	1.970	1.845
25	2.007	1.875	1.771
30	1.924	1.810	1.720
35	1.863	1.763	1.683
40	1.817	1.727	1.654
45	1.718	1.698	1.631
50	1.751	1.674	1.612
55	1.726	1.655	1.592
60	1.705	1.638	1.583
65	1.687	1.623	1.571
70	1.671	1.610	1.561
75	1.657	1.599	1.552
80	1.644	1.589	1.544
85	1.633	1.580	1.536
90	1.623	1.571	1.529
95	1.613	1.564	1.523
100	1.605	1.557	1.518
105	1.597	1.551	1.513
110	1.590	1.545	1.508
115	1.583	1.540	1.503
120	1.577	1.534	1.499
125	1.571	1.530	1.495
130	1.566	1.525	1.492
135	1.561	1.521	1.488
140	1.556	1.517	1.485
145	1.551	1.514	1.482
150	1.547	1.510	1.479
155	1.543	1.507	1.477
160	1.539	1.504	1.474
165	1.536	1.501	1.472
170	1.532	1.498	1.469
175	1.529	1.495	1.467
180	1.526	1.493	1.465
185	1.523	1.490	1.463
190	1.520	1.488	1.461
195	1.517	1.486	1.459
200	1.515	1.483	1.458

Critical Values for C_{pk}=1.50

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	3.159	2.774	2.496
15	2.635	2.394	2.213
20	2.397	2.215	2.076
25	2.257	2.109	1.993
30	2.163	2.036	1.936
35	2.096	1.984	1.894
40	2.044	1.943	1.862
45	2.004	1.911	1.836
50	1.970	1.884	1.815
55	1.942	1.862	1.798
60	1.919	1.843	1.782
65	1.898	1.827	1.769
70	1.880	1.813	1.758
75	1.865	1.800	1.747
80	1.851	1.789	1.738
85	1.838	1.779	1.730
90	1.827	1.769	1.722
95	1.816	1.761	1.716
100	1.807	1.753	1.709

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
105	1.798	1.746	1.704
110	1.790	1.740	1.698
115	1.782	1.734	1.693
120	1.776	1.728	1.689
125	1.769	1.723	1.685
130	1.763	1.718	1.680
135	1.757	1.713	1.677
140	1.752	1.709	1.673
145	1.747	1.705	1.670
150	1.742	1.701	1.667
155	1.738	1.697	1.664
160	1.733	1.694	1.661
165	1.729	1.690	1.658
170	1.726	1.687	1.655
175	1.722	1.684	1.653
180	1.718	1.681	1.651
185	1.715	1.679	1.648
190	1.712	1.676	1.646
195	1.709	1.674	1.644
200	1.706	1.671	1.642

Testing Procedure

$H_0: C_{pk} \le C$

$H_1: C_{pk} > C$

STEP 1: Decide the definition of 'capable' (set the value of C), and the α-risk (normally set to 0.01, 0.025, or 0.05), the chance of wrongly concluding an incapable process as capable.

STEP 2: Calculate the value of $\hat{C}_{pk}$ from the sample.

STEP 3: Check the appropriate table , and find the critical value c_0 based on C, α-risk and n.

STEP 4: Conclude that the process is capable ($C_{pk} > C$) if $\hat{C}_{pk}$ value is greater than the critical value c_0 ($\hat{C}_{pk} > c_0$). Otherwise, we do not have enough information to conclude that the process is capable.

A total of 60 observations collected from factory

USL set to 5μA,

LSL set to -5μA,

For these 60 sample measurements, under the Shapiro-Wilk test for normality, the statistic W is found to be 0.9656 with *p*-value 0.088.

That is,

it is reasonable to assume that the process data collected from the factory are normally distributed.

Table. A total of 60 observations collected from factory.

0.344	-0.804	1.413	1.565	-1.057	1.071	-0.342	-0.086
1.027	0.367	0.590	1.337	-0.251	0.551	-0.004	1.051
0.458	-0.120	-0.508	-0.225	-1.529	3.408	1.673	0.811
0.061	-0.283	0.921	-0.044	1.354	-1.006	-0.657	-0.316
-0.205	0.568	0.588	-0.620	-0.215	-0.017	1.344	-0.514
0.395	1.259	0.135	-0.372	-0.720	0.925	2.435	-1.141
0.812	0.432	-0.695	1.526	-0.559	-0.504	-1.692	-0.883
-0.670	0.518	2.612	-0.525				

An Example

Suppose for this particular process under their Six Sigma program, the process index must reach at least a certain level C, say, 1.50, i.e., the requirement for the process yield is no more than 3.4 PPM.

Based on $C = 1.50$, $\alpha = 0.05$, and $n = 60$, the critical value is found to be $c_0 = 1.7818$

The calculated sample mean, $\bar{x} = 0.250$ the sample standard deviation, $s = 1.011$.

Since $\hat{C}_{pk} = 1.566 < c_0$,

we may claim that the process is incapable with 95% of confidence

An Example

- On the other hand,

with the sample information,
we obtain the corresponding $\boxed{p\text{-value} = 0.351}$
and the lower confidence bound $\boxed{C_{pk}^{L} = 1.316}$

We therefore conclude that the true value of the process capability C_{pk} is no less than 1.316 with 95% level of confidence.

process yield no less than 99.9921%
with fraction of nonconformities about 79 PPM.

Comments

Therefore, in this case, the consumer would reject the entire product.

We note, however, that if the classical approach is applied it is almost certain that any sample of 60 products will contain no defective items.

In fact, the probability that the sample contains at least one defective item is less than $\boxed{0.006.}$

All the products therefore will be accepted, which obviously provides no quality assurance at all.

Estimation of C_{pm}

$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}} = \frac{USL-LSL}{6\sqrt{E[(X-T)^2]}}$$

- Estimation of C_{pm} (Chan, Cheng, Spiring (1988), **Boyles** (1991))

$$\tilde{C}_{pm} = \hat{C}_{pm(CCS)} = \frac{d}{3\sqrt{\sum_{i=1}^{n}(x_i-T)^2/(n-1)}} = \frac{d}{3\sqrt{s^2+\frac{n}{n-1}(\bar{x}-T)^2}}$$

$$\hat{\boldsymbol{C}}_{\boldsymbol{pm}} = \hat{\boldsymbol{C}}_{\boldsymbol{pm(B)}} = \frac{d}{3\sqrt{\sum_{i=1}^{n}(x_i-T)^2/n}} = \frac{d}{3\sqrt{s_n^2+(\bar{x}-T)^2}}$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$, $s^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$

and $s_n^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2/n = \frac{n-1}{n}s^2$

Sampling Distribution of the Estimated C_{pm}

Boyles (1991) and Pearn *et al.* (1992) have shown that the distributed of $\hat{C}_{pm}$ is

$$\hat{C}_{pm} \sim C_{pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\frac{n}{\chi'^2_{n,\lambda}}},$$

where, as above, $\chi'^2_{n,\lambda}$ denotes the non-central chi-square distribution with n degrees of freedom and the non-centrality parameter $\lambda = n\xi^2$, $\xi = (\mu - T)/\sigma$.

Sampling Distribution of the Estimated C_{pm}

The p.d.f. of $\hat{C}_{pm}$ (Chen *et al.* (1999))

$$f_{\hat{C}_{pm}}(y) = \frac{2^{1-n/2} D^n}{3^n y^{n+1}} \exp\left(-\frac{\lambda}{2} - \frac{D^2}{18y^2} \right) \times$$

$$\sum_{j=0}^{\infty} \left\{ \left(\frac{\lambda D^2}{36y^2} \right)^j \Big/ \left(j!\,\Gamma\left(\frac{n}{2} + j \right) \right) \right\},\ y > 0,$$

which follows a non-central chi-square distribution with 1 degree of freedom and the non-centrality parameter $\lambda = \delta^2 = n(\mu - T)^2 / \sigma^2$).

Approximate Confidence Interval for C_{pm}

Marcucci and Beazley (1988) $C_{pm}^{L(MB)} \cong \sqrt{\frac{\chi_n^2(1-\gamma)}{n}}\hat{C}_{pm}$

Boyles (1991) $C_{pm}^{L(Bo)} \cong \hat{C}_{pm}\sqrt{\frac{\chi_{\hat{\nu}}^2(1-\gamma)}{\hat{\nu}}}$,

where $\hat{\nu} = \frac{n\left(1+\hat{\xi}^2\right)^2}{\left(1+2\hat{\xi}^2\right)}$, $\hat{\xi} = \frac{(\bar{X}-T)}{S}$.

Zimmer and Hubele (1997): $C_{pm}^{L(ZH)} = \hat{C}_{pm}\sqrt{\frac{\chi_{n,\lambda}^2(1-\gamma)}{n+\lambda}}$.

Sampling Distribution of the Estimated C_{pm}

The c.d.f. of $\hat{C}_{pm}$ is expressed in terms of a mixture of the chi-square distributions and a standard normal distribution (Pearn and Lin (2003))

$$F_{\hat{C}_{pm}}(y) = 1 - \int_0^{b\sqrt{n}/(3y)} G\left(\frac{b^2 n}{9y^2} - t^2\right) \times \left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right] dt,$$

for $y > 0$, where $b = d/\sigma$, and as above $\xi = (\mu - T)/\sigma$, $G(\cdot)$ is the c.d.f. of the chi-square distribution with degree of freedom $n-1$, χ^2_{n-1}, and $\phi(\cdot)$ is the p.d.f. of the standard normal variable.

Capability Testing Based on C_{pm}

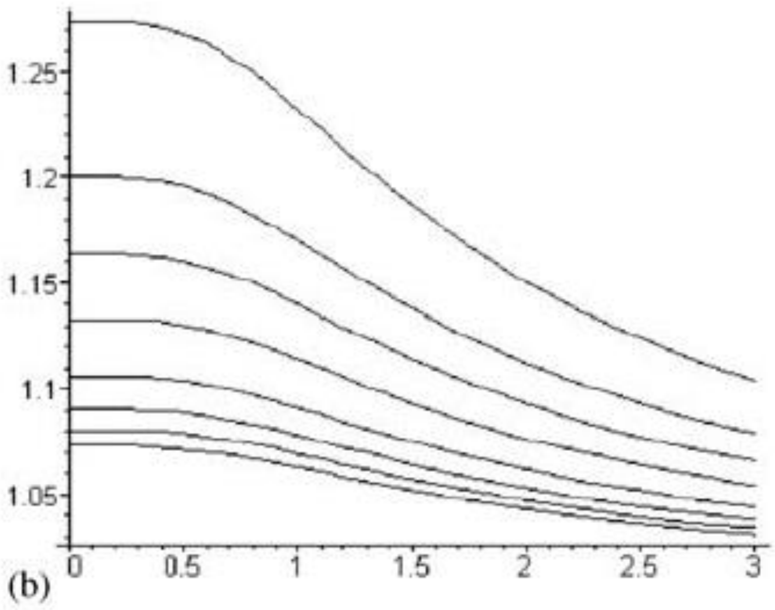
H_0: $C_{pm} \le C$ (process is not capable),
H_1: $C_{pm} > C$ (process is capable).

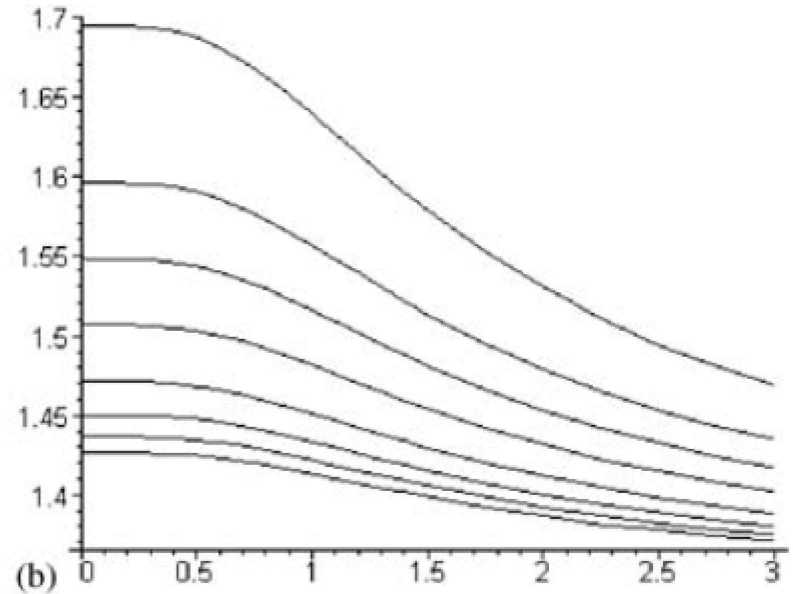
$$\boxed{P(\hat{C}_{pm} \ge c_0 | C_{pm} = C) = \alpha.}$$

given capability requirement C, parameter ξ, sample size n, and risk α, the critical value c_0 can be obtained by solving the following equation:

$$\boxed{\int_0^{b\sqrt{n}/(3c_0)} G\left(\frac{b^2 n}{9c_0^2} - t^2\right)\left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right] dt = \alpha.}$$

Critical Value vs Parameter ξ=(μ-*T*)/σ





C_{pm}=1.00 and C_{pm}=1.33 for various sample size

Lower Confidence Bound on C_{pm}

We note that when $C_{pm} = C$, the term $b = d/\sigma$ can be expressed as $b = 3C\sqrt{1+\xi^2}$. Thus, the index C_{pm} may be expressed as a function of the distribution characteristic parameter .

$$C_{pm} = \frac{d}{3\sigma\sqrt{1+\xi^2}} = \frac{d/\sigma}{3\sqrt{1+\xi^2}}.$$

Hence, given the sample of size n, the confidence level γ, the estimated value, $\hat{C}_{pm}$, and the parameter ξ, the lower confidence bounds (denoted as $C_{pm}^{L(PS)}$ can be obtained using a numerical integration technique with iterations, to solve the following

$$\boxed{\int_0^{b\sqrt{n}/(3\hat{C}_{pm})} F_K\left(\frac{b^2 n}{9\hat{C}_{pm^2}} - t^2\right)[\phi(\mathrm{t}+\xi\sqrt{n}) + \phi(\mathrm{t}-\xi\sqrt{n})]\mathrm{d}t = 1-\gamma.}$$

P-Value

For processes with target value setting on the middle of the specification limits ($T = m = (USL + LSL)/2$), which are fairly common situations, the index may be rewritten as: $C_{pm} = b/[3(1+\xi^2)^{1/2}]$. Given $C_{pm} = C$, $b = d/\sigma$ can be expressed as $b = 3C(1+\xi^2)^{1/2}$. Given a value of C (the capability requirement), the *p*-value corresponding to c^*, a specific value of $\hat{C}_{pm}$ calculated from the sample data, is :

$$\boxed{\begin{aligned} &P(\hat{C}_{pm} \geq c^* | C_{pm} = C) = \\ &\int_0^{b\sqrt{n}/(3c^*)} G\left(\frac{b^2 n}{9(c^*)^2} - t^2\right) \left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right] dt. \end{aligned}}$$

Critical Values for C_{pm}=1.00

n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$	n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$
10	1.978	1.755	1.594	210	1.128	1.106	1.088
15	1.694	1.548	1.438	215	1.126	1.105	1.087
20	1.557	1.445	1.358	220	1.124	1.104	1.086
25	1.473	1.381	1.309	225	1.123	1.102	1.085
30	1.417	1.337	1.274	230	1.121	1.101	1.084
35	1.376	1.305	1.249	235	1.120	1.100	1.083
40	1.344	1.280	1.229	240	1.119	1.099	1.082
45	1.319	1.260	1.213	245	1.117	1.098	1.081
50	1.298	1.244	1.200	250	1.116	1.097	1.080
55	1.280	1.230	1.189	255	1.115	1.096	1.080
60	1.266	1.218	1.179	260	1.113	1.095	1.079
65	1.253	1.208	1.171	265	1.112	1.094	1.078
70	1.242	1.199	1.164	270	1.111	1.093	1.077
75	1.232	1.191	1.157	275	1.110	1.092	1.076
80	1.223	1.184	1.151	280	1.109	1.091	1.076
85	1.215	1.177	1.146	285	1.108	1.090	1.075
90	1.208	1.171	1.142	290	1.107	1.089	1.074
95	1.201	1.166	1.137	295	1.106	1.088	1.074
$\boxed{100}$	$\boxed{1.195}$	$\boxed{1.161}$	$\boxed{1.133}$	300	1.105	1.087	1.073
105	1.190	1.157	1.130	305	1.104	1.087	1.072
110	1.185	1.153	1.126	310	1.103	1.086	1.072
115	1.180	1.149	1.123	315	1.102	1.085	1.071
120	1.175	1.145	1.120	320	1.101	1.084	1.070
125	1.171	1.142	1.118	325	1.100	1.84	1.070
130	1.168	1.139	1.115	330	1.100	1.083	1.069
135	1.164	1.136	1.113	335	1.099	1.082	1.069
140	1.161	1.133	1.110	340	1.098	1.82	1.068
145	1.157	1.131	1.108	345	1.097	1.081	1.068
150	1.154	1.128	1.106	350	1.096	1.081	1.067
155	1.151	1.126	1.104	355	1.096	1.080	1.067
160	1.149	1.124	1.102	360	1.095	1.079	1.066
165	1.146	1.121	1.101	365	1.094	1.079	1.066
170	1.144	1.119	1.099	370	1.094	1.078	1.065
175	1.141	1.117	1.098	375	1.093	1.078	1.065
180	1.139	1.116	1.096	380	1.092	1.077	1.064
185	1.137	1.114	1.095	385	1.092	1.077	1.064
190	1.135	1.112	1.093	390	1.091	1.076	1.063
195	1.133	1.111	1.092	395	1.090	1.075	1.063
200	1.131	1.109	1.091	400	1.090	1.075	1.063
205	1.129	1.108	1.089	405	1.089	1.074	1.062

Estimation of C_{pmk}

$$C_{pmk} = \min\left\{ \frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}} \right\} = \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}}$$

- Estimation of C_{pmk}

$$\hat{C}_{pmk} = \min\left\{ \frac{USL-\bar{x}}{3\sqrt{s_n^2+(\bar{x}-T)^2}}, \frac{\bar{x}-LSL}{3\sqrt{s_n^2+(\bar{x}-T)^2}} \right\}$$

$$= \frac{d-|\bar{x}-M|}{3\sqrt{s_n^2+(\bar{x}-T)^2}}$$

Sampling Distribution of the Estimated C_{pmk}

$\hat{C}_{pmk}$ is distributed as a mixture of the chi-square and the non-central chi-square variables expressed as (Pearn *et al.* (1992)):

$$\hat{C}_{pmk} \sim \frac{\dfrac{d\sqrt{n}}{\sigma} - \chi'_{1,\lambda}}{3\sqrt{\chi^2_{n-1} + \chi'^2_{1,\lambda}}}$$

with $\chi'^2_{1,\lambda}$ is a non-central chi-squared variable with one degree of freedom and the non-centrality parameter $\lambda = n(\mu - T)^2 / \sigma^2$.

Sampling Distribution of the Estimated C_{pmk}

The p.d.f. of $\hat{C}_{pmk}$ (Pearn *et al.* (2001))

$$f_{\hat{C}_{pmk}}(y)=\begin{cases} \displaystyle\int_1^{\infty} f_K\left(\frac{\left(D-\sqrt{S_D(y)t}\right)^2}{9y^2}-S_D(y)t\right)\times \\ \displaystyle f_H\left(S_D(y)t\right)\frac{2S_D(y)\left(D-\sqrt{S_D(y)t}\right)^2}{-9y^3}dt, \quad -\frac{1}{3}<y<0 \\ \displaystyle\int_0^{1} f_K\left(\frac{\left(D-\sqrt{S_D(y)t}\right)^2}{9y^2}-S_D(y)t\right)\times \\ \displaystyle f_H\left(S_D(y)t\right)\frac{2S_D(y)\left(D-\sqrt{S_D(y)t}\right)^2}{-9y^3}dt, \quad y>0, \end{cases}$$

where $S_D(y)=[D/(1+3y)^2]$, $F_K(\cdot)$ is the c.d.f. of K (a χ^2_{n-1} variable), $f_K(\cdot)$ is the p.d.f. of K, and $f_H(\cdot)$ is the probability density function of H (a non-central χ^2 variable).

Approximate Confidence Interval for C_{pmk}

An asymptotic $100(1-\alpha)\%$ confidence interval on C_{pmk} derived by Chen and Hsu (1995):

$$\left[\hat{C}_{pmk} - z_{\alpha/2}\frac{\hat{\sigma}_{pmk}}{\sqrt{n}},\ \hat{C}_{pmk} + z_{\alpha/2}\frac{\hat{\sigma}_{pmk}}{\sqrt{n}}\right]$$

where

$$\hat{\sigma}^2_{pmk} = \left(\frac{1}{9(1+\delta^2)} + \frac{2\delta}{3(1+\delta^2)^{3/2}}\right)\hat{C}_{pmk} + \frac{72\delta^2 + d\left(\dfrac{m_4}{s_n^4} - 1\right)}{72(1+\delta^2)^2}\hat{C}^2_{pmk},$$

$$m_4 = \sum_{i=1}^{n}(x_i - \bar{x})^4/n\ ,\ \delta = (\bar{x} - T)/s_n\ ,$$

$$s_n = \sum_{i=1}^{n}(x_i - \bar{x})^2/n\ .$$

Sampling Distribution of the Estimated C_{pmk}

The c.d.f. of $\hat{C}_{pmk}$ can be expressed as

$$F_{\hat{C}_{pmk}}(y) = 1 - \int_0^{b\sqrt{n}/(1+3y)} G\left(\frac{(b\sqrt{n}-t)^2}{9y^2} - t^2\right) \times \left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right] dt,$$

for $y > 0$, where $b = d/\sigma$, $\xi = (\mu - T)/\sigma$, $G(\cdot)$ is the c.d.f. of a chi-squared random variable with $n-1$ degrees of freedom. χ^2_{n-1} and $\phi(\cdot)$ is the p.d.f. of the standard normal variable $N(0,1)$ (see Pearn and Lin (2002)).

Capability Testing Based on C_{pmk}

H_0: $C_{pmk} \leqslant C$ (process is not capable),

H_1: $C_{pmk} > C$ (process is capable).

$$\alpha = P(\hat{C}_{pmk} \geqslant c_0 \mid C_{pmk} = C)$$

given capability requirement C, parameter ξ, sample size n, and risk α,

the critical value c_0 can be obtained by solving the following equation:

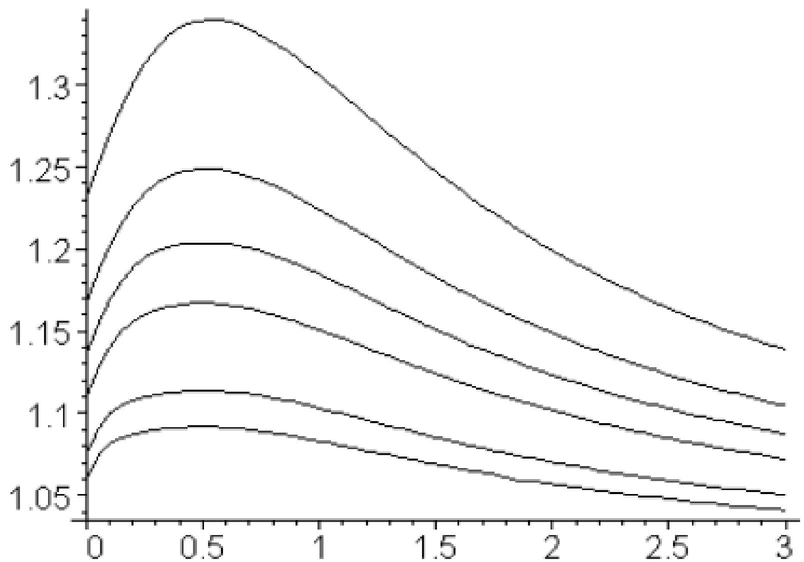
$$\boxed{\alpha = \int_0^{b\sqrt{n}/(1+3c_0)} G\left(\frac{(b\sqrt{n}-t)^2}{9c_0^2} - t^2\right)[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})]\,\mathrm{d}t}$$

P-Value

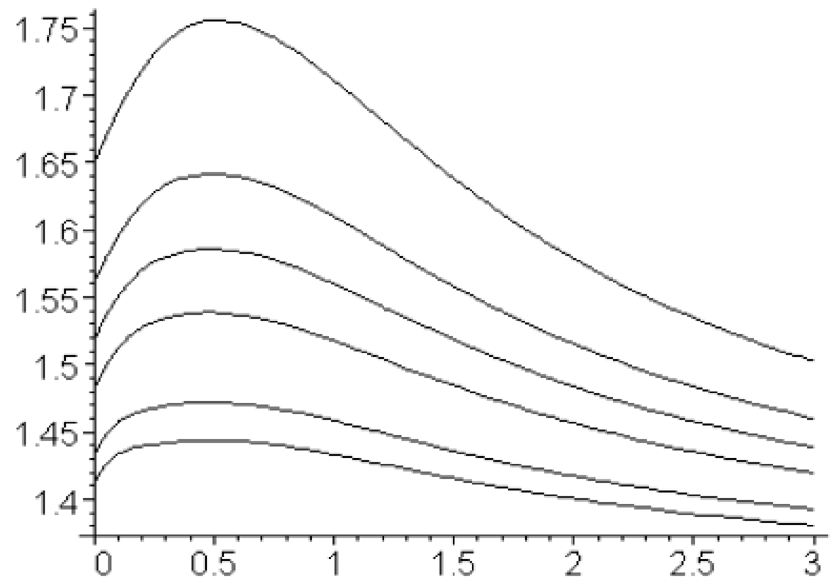
$$p\text{-value} = P\{\hat{C}_{pmk} \geqslant c^* \mid C_{pmk} = C\}$$

$$= \int_0^{b\sqrt{n}/(1+3c^*)} G\left(\frac{(b\sqrt{n}-t)^2}{9(c^*)^2} - t^2\right)[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})]\,\mathrm{d}t$$

Critical Value vs Parameter $\xi=(\mu-T)/\sigma$



Plots of c_0 versus $|\xi|$ for $C_{pmk} = 1.00$, $\alpha = 0.05$, $n = 30, 50, 70, 100\ 200, 300$ (top to bottom in plot)



Plots of c_0 versus $|\xi|$ for $C_{pmk} = 1.33$, $\alpha = 0.05$, and $n = 30, 50, 70, 100\ 200, 300$ (top to bottom in plot)

Estimations of C_{PU} and C_{PL}

- Estimations of C_{PU} and C_{PL}

$$C_{PU} = \frac{USL - \mu}{3\sigma} \qquad C_{PL} = \frac{\mu - LSL}{3\sigma}$$

$$\hat{C}_{PU} = \frac{USL - \bar{x}}{3s} \qquad \hat{C}_{PL} = \frac{\bar{x} - LSL}{3s}$$

- Pearn and Chen (2002) obtained the **UMVUEs** of C_{PU} and C_{PL} by adding the **correction factor** to $\hat{C}_{PU}$ and $\hat{C}_{PL}$ as

$$\tilde{C}_{PU} = b_{n-1}\hat{C}_{PU}$$

$$\tilde{C}_{PL} = b_{n-1}\hat{C}_{PL} \qquad b_{n-1} = \sqrt{\tfrac{2}{n-1}}\,\Gamma\left(\tfrac{n-1}{2}\right)/\Gamma\left(\tfrac{n-2}{2}\right)$$

Sampling Distribution of the Estimated C_{PU} and C_{PL}

Under the normality assumption (Chou and Owen (1989)),

$$\hat{C}_{PU} \sim (3\sqrt{n})^{-1}\, t_{n-1}(\delta),$$

where $t_{n-1}(\delta)$ is a non-central t random variable with $n-1$ degrees of freedom and the non-centrality parameter $\delta = 3\sqrt{n}\, C_{PU}$.

The estimator $\hat{C}_{PL}$ has the same sampling distribution as $\hat{C}_{PU}$ with $\delta = 3\sqrt{n}\, C_{PL}$.

Sampling Distribution of the Estimated C_{PU} and C_{PL}

The p.d.f. of $\tilde{C}_I$ can be expressed as

$$f(x) = \frac{3\sqrt{n/(n-1)} \cdot 2^{-n/2}}{b_{n-1}\sqrt{\pi}\,\Gamma[(n-1)/2]} \times \int_0^\infty y^{(n-2)/2} \exp\left\{-\frac{1}{2}\left[y + \left(\frac{3x\sqrt{ny}}{b_{n-1}\sqrt{n-1}} - \delta\right)^2\right]\right\} dy,$$

where $b_{n-1} = [2/(n-1)]^{1/2}\,\Gamma[(n-1)/2]/\Gamma[(n-2)/2]$ and $\delta = 3\sqrt{n}\,C_I$.

PDF plots of the Estimated C_{PU} and C_{PL}

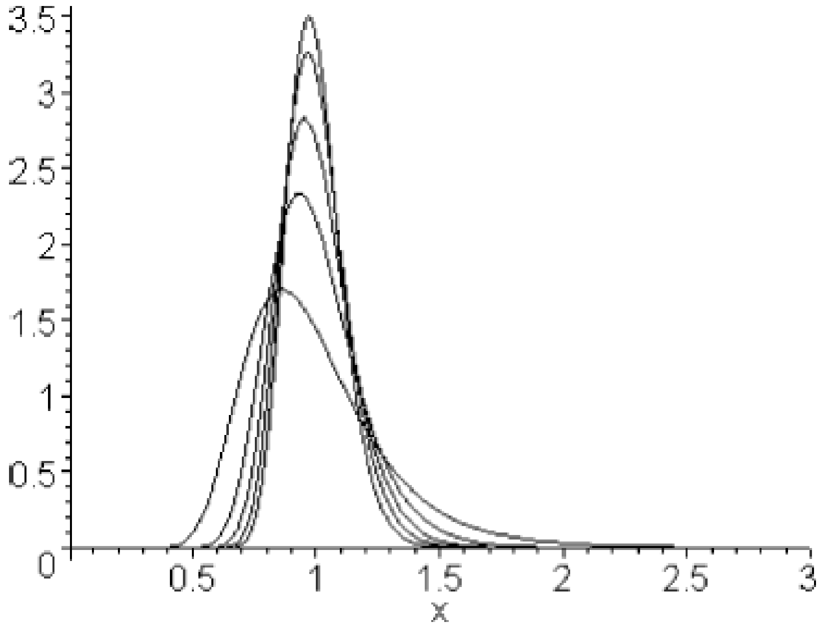


Fig. 1. PDF plot of $\tilde{C}_{PU}$ for $C_{PU} = 1.00$ and $n = 10, 20, 30, 40, 50$ (from bottom to top in plot).

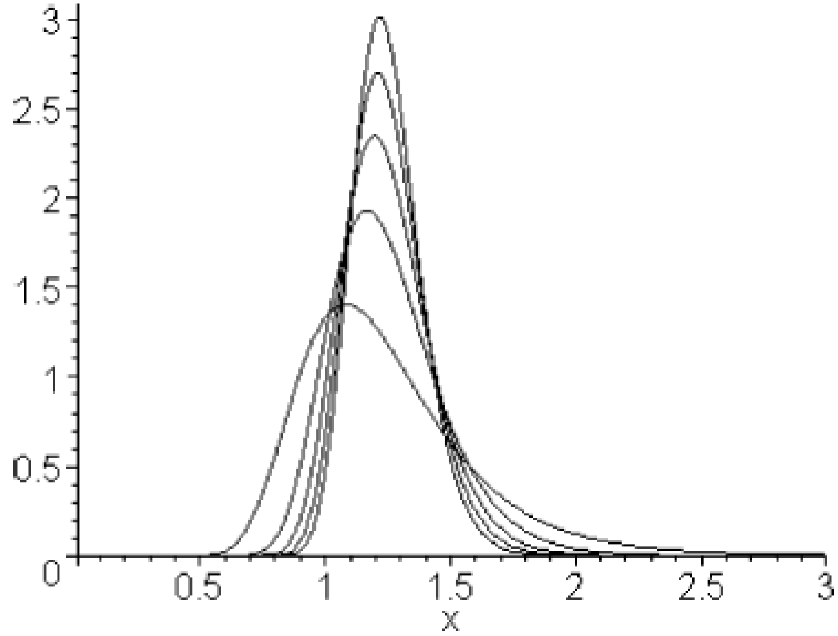


Fig. 2. PDF plot of $\tilde{C}_{PU}$ for $C_{PU} = 1.25$ and $n = 10, 20, 30, 40, 50$ (from bottom to top in plot).

Capability Testing Based on C_{PU} and C_{PL}

$$H_0:\ C_{\mathrm{PU}} \leqslant C \qquad\qquad H_0:\ C_{\mathrm{PL}} \leqslant C$$

$$H_1:\ C_{\mathrm{PU}} > C \qquad\qquad H_1:\ C_{\mathrm{PL}} > C$$

$$\alpha = P\left\{\widetilde{C}_{\mathrm{PU}} \geqslant c_0 | C_{\mathrm{PU}} = C\right\}$$

$$= P\left\{3\sqrt{n}\widehat{C}_{\mathrm{PU}} \geqslant \frac{3\sqrt{n}c_0}{b_{n-1}} | C_{\mathrm{PU}} = C\right\}$$

$$= P\left\{t_{n-1}(\delta) \geqslant \frac{3\sqrt{n}c_0}{b_{n-1}}\right\}$$

$$\Downarrow$$

$$t_{n-1,\alpha}(\delta)$$

$$c_0 = b_{n-1}t_{n-1,\alpha}(\delta)/(3\sqrt{n}),$$

where $\delta = 3\sqrt{n}C$.

Critical values c_0 for $C = 1.25$, $n = 10(5)505$ and $\alpha = 0.01, 0.025, 0.05$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	2.420	2.124	1.910
15	2.087	1.895	1.750
20	1.927	1.780	1.667
25	1.830	1.708	1.614
30	1.763	1.659	1.576
35	1.715	1.622	1.548
40	1.677	1.593	1.526
45	1.647	1.570	1.508
50	1.622	1.551	1.493
55	1.602	1.535	1.481
60	1.584	1.521	1.470
65	1.568	1.509	1.460
70	1.555	1.498	1.452
75	1.543	1.489	1.444
80	1.532	1.480	1.438
85	1.522	1.472	1.432
90	1.513	1.465	1.426
95	1.505	1.459	1.421
100	1.498	1.453	1.416

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
105	1.491	1.448	1.412
110	1.485	1.443	1.408
115	1.479	1.438	1.404
120	1.474	1.434	1.401
125	1.469	1.430	1.398
130	1.464	1.426	1.394
135	1.460	1.422	1.392
140	1.455	1.419	1.389
145	1.451	1.416	1.386
150	1.448	1.413	1.384
155	1.444	1.410	1.382
160	1.441	1.407	1.379

Applications of PCIs

Acceptance Sampling Plans Based on ***C**pk*

Designing a Sampling Plan

- The acceptance sampling plan must have its OC curve passing through those two designated points **(AQL, 1-α)** and **(LTPD, β)**.

$$\Pr\{\text{Accepting the lot} \mid p = \text{AQL}\} \geq 1 - \alpha$$

$$\Pr\{\text{Accepting the lot} \mid p = \text{LTPD}\} \leqslant \beta$$

- **AQL**: Acceptable Quality Level
- **LTPD**: Lot Tolerance Percent Defective,
  also called **RQL** (Rejectable Quality Level)

Nomograph for Designing **Attributes** Sampling Plans

$$1-\alpha = \sum_{d=0}^{c} \frac{n!}{d!(n-d)!} p_1^d (1-p_1)^{n-d}$$

$$\beta = \sum_{d=0}^{c} \frac{n!}{d!(n-d)!} p_2^d (1-p_2)^{n-d}$$

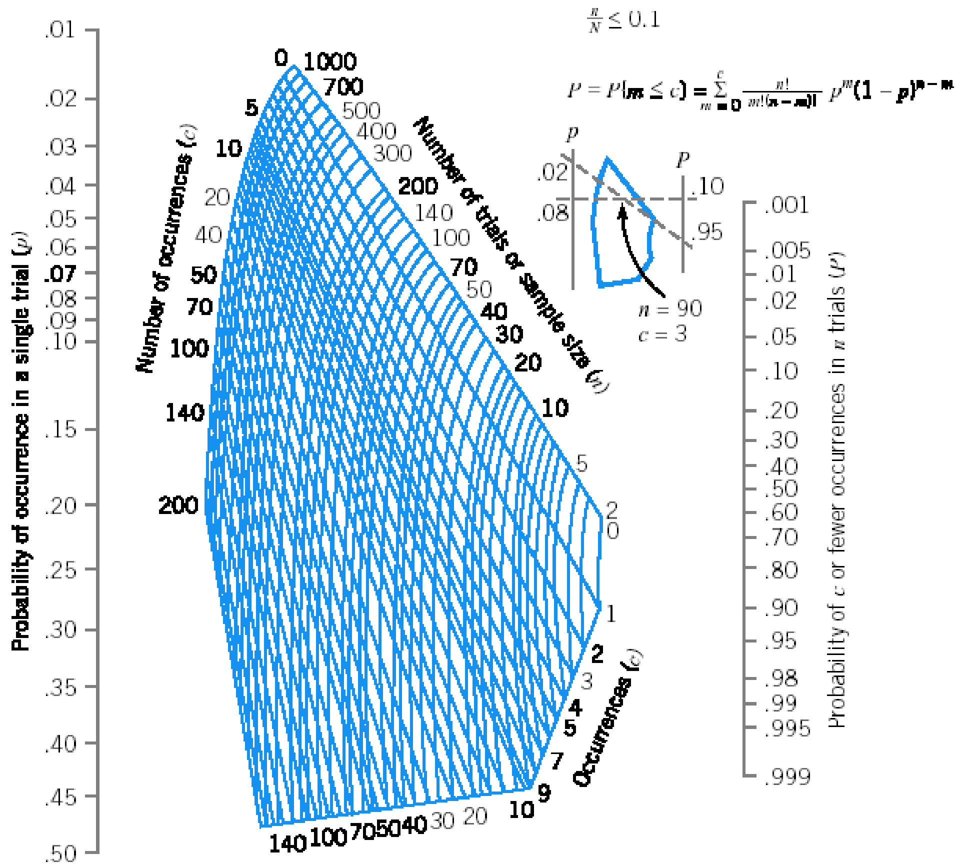


Figure 14-9 Binomial nomograph.

Example 1:

Given p_{AQL}= 10 PPM and p_{RQL} = 1000 PPM with α-risk = 0.05 and β-risk = 0.10

➔ (n, c)=(2302, 0)

Example 2:

Given p_{AQL}= 10 PPM and p_{RQL} = 100 PPM with α-risk = 0.05 and β-risk = 0.10

➔ (n, c)=(53224, 2)

Nomograph for Designing **Variables** Sampling Plans

$$Z_{\mathrm{USL}} = \frac{\mathrm{USL} - \bar{x}}{\sigma}$$

$$Z_{\mathrm{LSL}} = \frac{\bar{x} - \mathrm{LSL}}{\sigma}$$

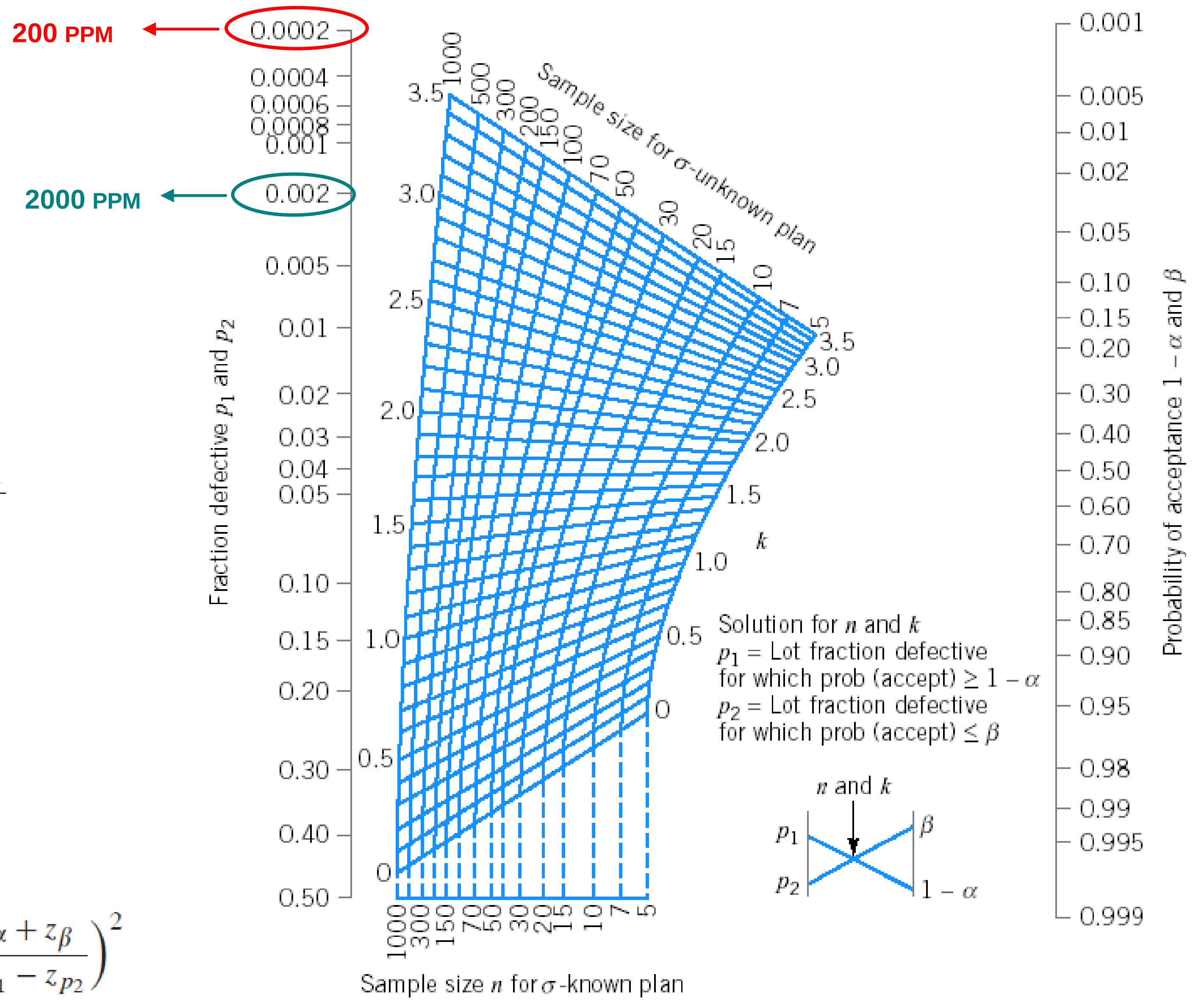
$$k = \frac{z_\alpha z_{p_2} + z_\beta z_{p_1}}{z_\alpha + z_\beta}$$

if σ is known,

$$n = \left(\frac{z_\alpha + z_\beta}{z_{p_1} - z_{p_2}} \right)^2$$

if σ is unknown,

$$n = \left(1 + \frac{k^2}{2}\right) \left(\frac{z_\alpha + z_\beta}{z_{p_1} - z_{p_2}} \right)^2$$



Designing a Sampling Plan based on C_{pk}

$$\boxed{C_{pk} = C_{\mathrm{AQL}}}$$

$$\Uparrow$$

$$\Pr\{\text{Accepting the lot} \mid p = \mathrm{AQL}\} \geqslant 1 - \alpha$$

$$\Pr\{\text{Accepting the lot} \mid p = \mathrm{LTPD}\} \leqslant \beta$$

$$\Downarrow$$

$$\boxed{C_{pk} = C_{\mathrm{LTPD}}}$$

Designing a Sampling Plan based on Cpk

- **The cumulative distribution function of the estimated C_{pk}:**

$$F_{\hat{C}_{pk}}(y) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9ny^2}\right)[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})]\,\mathrm{d}t$$

- **The probability of accepting the product can be expressed as**

$$\begin{aligned}\pi_A(C_{pk}) &= P(\hat{C}_{pk} \geqslant c_0 | C_{pk}) \\ &= \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n}-t)^2}{9nc_0^2}\right)(\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n}))\,\mathrm{d}t.\end{aligned}$$

where $b = 3C_{pk} + |\xi|$, $\xi = (\mu - M)/\sigma$, $G(\cdot)$ is the CDF of χ^2_{n-1}, and $\phi(\cdot)$ is the PDF of $N(0, 1)$.

Designing a Sampling Plan based on C_{pk}

- The plan parameters (n, c_0) should be satisfied the following two conditions:

$$\int_0^{b_1\sqrt{n}} G\left(\frac{(n-1)(b_1\sqrt{n}-t)^2}{9nc_0^2}\right)(\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n}))\,\mathrm{d}t \geqslant 1-\alpha$$

$$\int_0^{b_2\sqrt{n}} G\left(\frac{(n-1)(b_2\sqrt{n}-t)^2}{9nc_0^2}\right)(\phi(t+\xi\sqrt{n})+\phi(t-\xi\sqrt{n}))\,\mathrm{d}t \leqslant \beta$$

where $b_1 = 3C_{\mathrm{AQL}} + |\xi|$ and $b_2 = 3C_{\mathrm{LTPD}} + |\xi|$, $C_{\mathrm{AQL}} > C_{\mathrm{LTPD}}$.

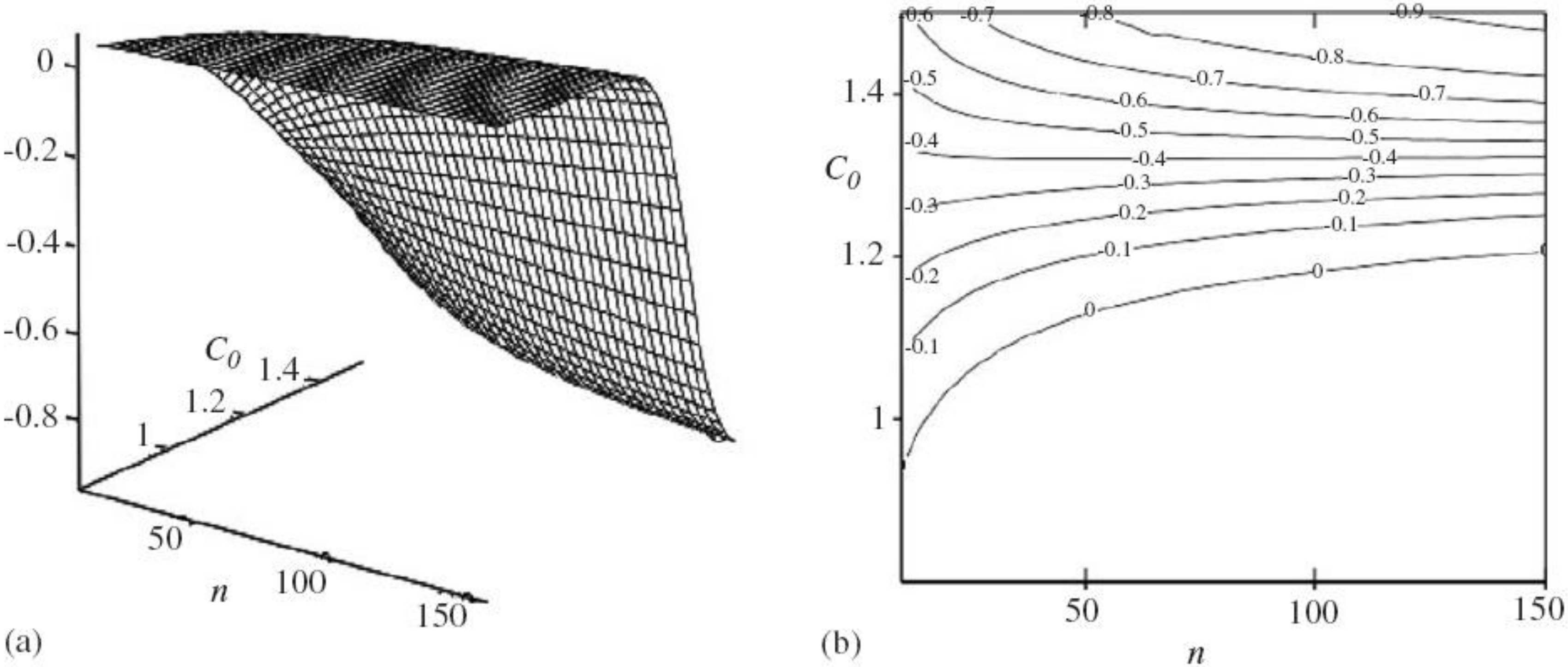
An Illustrative Example

- In the contract, the C_{AQL} and C_{LTPD} are set to 1.33 and 1.00 with α-risk = 0.05 and β-risk = 0.10.
- That is,

$$\Pr\{\text{Accepting the lot} \mid \boxed{\begin{array}{c} p = 66\text{PPM} \\ C_{\text{AQL}} = 1.33 \end{array}}\} \geq 0.95$$

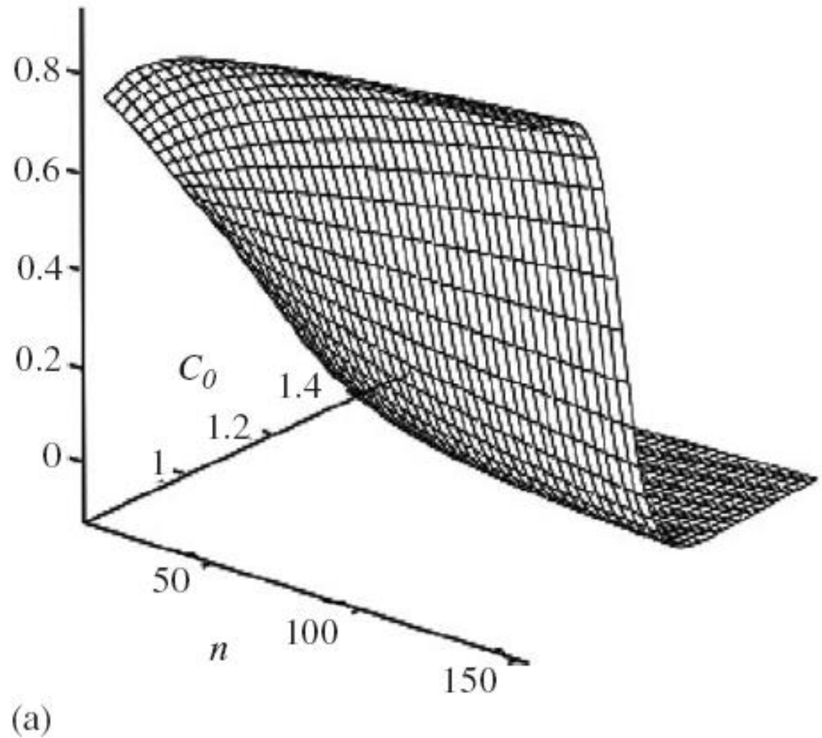
$$\Pr\{\text{Accepting the lot} \mid \boxed{\begin{array}{c} p = 2700\text{PPM} \\ C_{\text{LTPD}} = 1.00 \end{array}}\} \leq 0.10$$

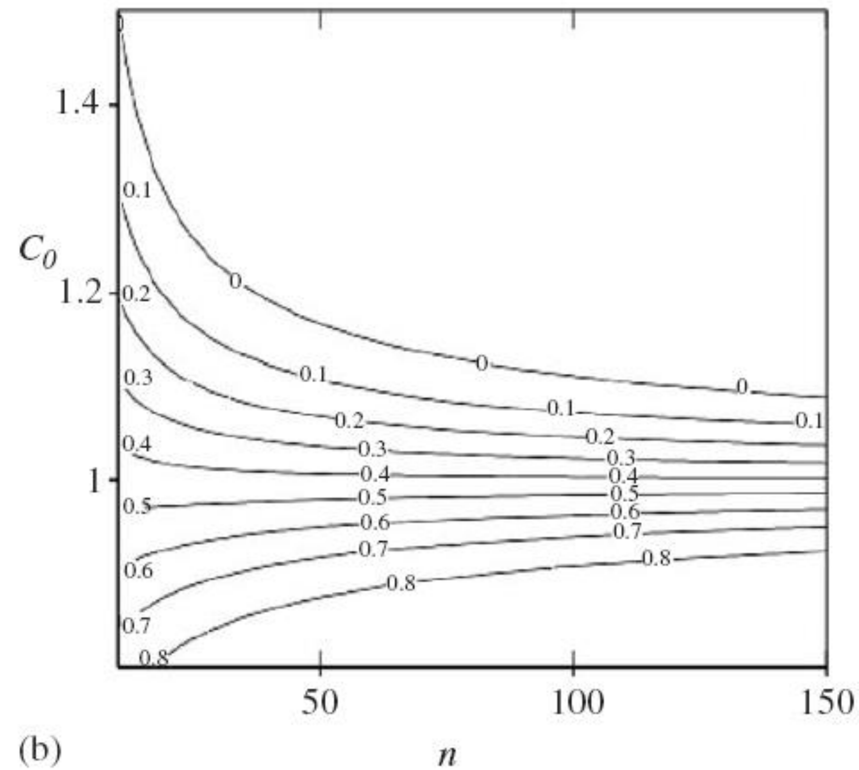
$$S_1(n, c_0) = \int_0^{b_1\sqrt{n}} G\left(\frac{(n-1)(b_1\sqrt{n}-t)^2}{9nc_0^2}\right) \times \left(\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right) \mathrm{d}t - (1-\alpha),$$



(a) Surface plot of $S_1(n, c_0)$. (b) Contour plot of $S_1(n, c_0)$.

$$S_2(n, c_0) = \int_0^{b_2\sqrt{n}} G\left(\frac{(n-1)(b_2\sqrt{n}-t)^2}{9nc_0^2}\right) \times (\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n}))\,\mathrm{d}t - \beta$$



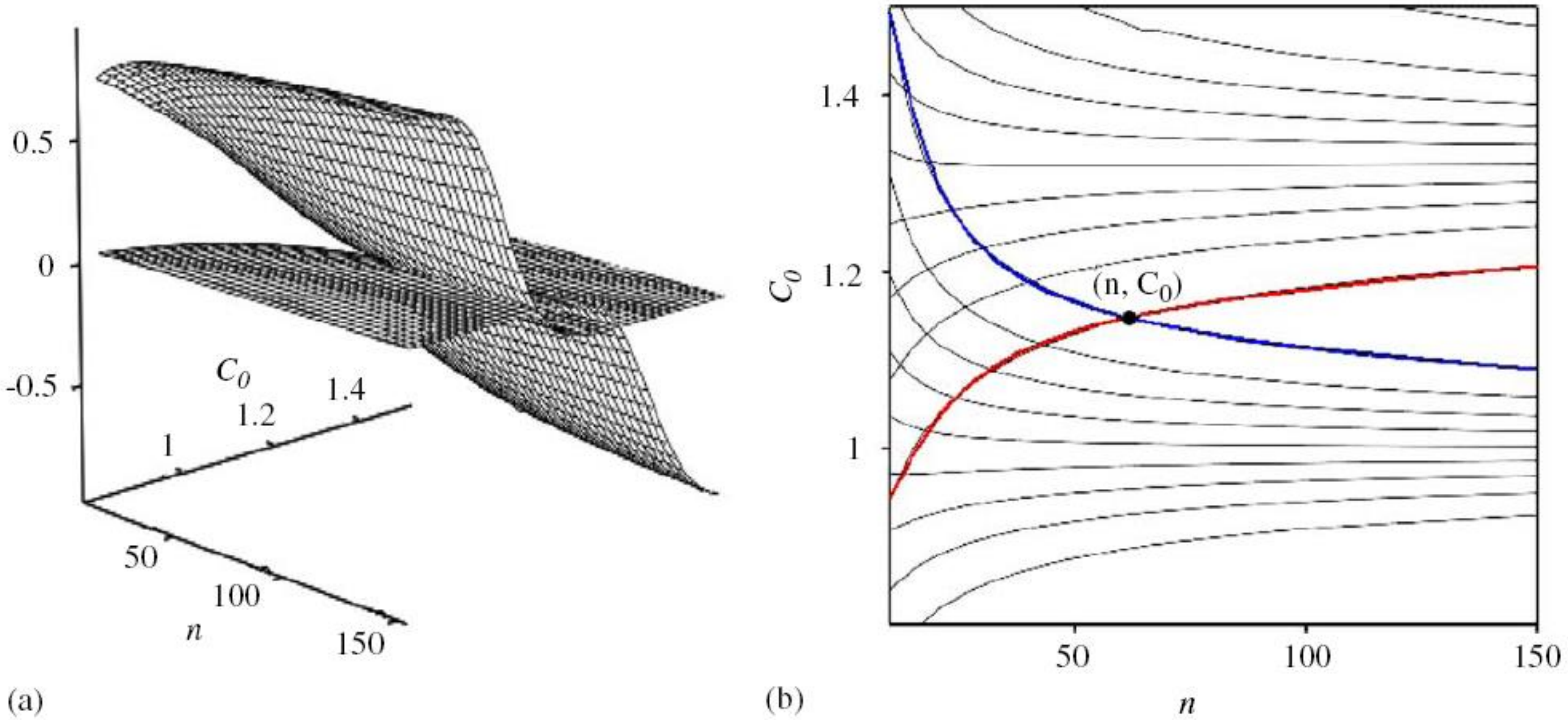


(a) Surface plot of $S_2(n, c_0)$. (b) Contour plot of $S_2(n, c_0)$.

Solution (n, C_0):

the interaction of S_1 and S_2 contour curves at level 0

(n, C_0)=(62, 1.1477)



(a) Surface plot of S_1 and S_2. (b) Contour plot of S_1 and S_2.

Required **Sample Size (*n*)** and **Critical Acceptance Values (c_0)** for various α and β risks with selected C_{AQL} and C_{LTPD}

α	β	$C_{AQL} = 1.33$, $C_{LTPD} = 1.00$		$C_{AQL} = 1.50$, $C_{LTPD} = 1.33$		$C_{AQL} = 1.67$, $C_{LTPD} = 1.33$		$C_{AQL} = 2.00$, $C_{LTPD} = 1.67$	
		n	c_0	n	c_0	n	c_0	n	c_0
0.01	0.01	158	1.1645	834	1.4149	232	1.4994	358	1.8345
	0.025	132	1.1509	704	1.4077	195	1.4853	301	1.8207
	0.05	112	1.1372	600	1.4005	166	1.4712	256	1.8069
	0.075	100	1.1271	537	1.3952	148	1.4607	229	1.7967
	0.10	91	1.1186	491	1.3907	135	1.4519	209	1.7881
0.025	0.01	136	1.1790	713	1.4222	200	1.5144	307	1.8489
	0.025	113	1.1655	593	1.4151	165	1.5033	254	1.8352
	0.05	94	1.1517	498	1.4078	139	1.4860	213	1.8212
	0.075	83	1.1414	441	1.4024	122	1.4754	188	1.8108
	0.10	75	1.1327	399	1.3977	110	1.4663	170	1.8019
0.05	0.01	119	1.1937	616	1.4297	174	1.5294	266	1.8635
	0.025	97	1.1805	505	1.4227	142	1.5157	218	1.8501
	0.05	80	1.1669	418	1.4154	117	1.5016	180	1.8362
	0.075	70	1.1566	366	1.4099	102	1.4908	157	1.8257
	0.10	62	1.1477	328	1.4052	91	1.4816	141	1.8167

An Illustrative Example (cont.)

- Consider USL= 20 mm, LSL= 10 mm, and the target value is set to $T = M = 15$.
- The required sample size and the acceptance critical value of sampling plan based on C_{AQL}= 1.33 and C_{LTPD} = 1.00 with α = 0.05 and β= 0.10 are obtained as (n, c_0) = **(62, 1.1447)**.

- Based on the collected sample of size n = 62, we obtain that Cpkhat =1.093.
- Thus, the consumer would "**reject**" the entire products since the sample estimator from the inspection, 1.093, is smaller than the critical acceptance value 1.1447.

Comments on Traditional Methods

- We note that if existing sampling plans are applied here, it is almost certain that any sample of 62 samples taken from the process will contain **zero** defective items. All the products therefore will be **accepted**, which obviously provide *no protection to the buyer* at all.

Conclusions

- Acceptance sampling plans provide the buyer and the vendor a decision rule for product sentencing to meet their needs.
- Since the sampling cannot guarantee that the defective items in a lot will be sampled and inspected, then the sampling involves risks of not adequately reflecting the quality conditions of the lot. Such a risk is even more significant as rapid advancement of the manufacturing technology and stringent customers demand is enforced.
- A sampling plan based on the index C_{pk} is developed to deal with product acceptance problem. It can be applied to determine the sample size required for inspection and the corresponding acceptance criterion, to provide the desired levels of protection to both producers and consumers.

Recommendations

- We remark again that these indices presented in the dissertation, are designed to monitor the performance for only **normal** and **near-normal** processes with **symmetric tolerances** ($T = M = (USL+LSL)/2$), which are shown to be inappropriate for cases with asymmetric tolerances.

- PCIs can be used only after it has been established that the manufacturing process is **under statistical control (stable)**.

- **Incorrectly applied and/or interpreted**, these indices can generate an abundance of misinformation that will confuse practitioners, waste resources, and lead to **make decision incorrectly**.

Future Research Directions

- For non-normal processes
- For asymmetric tolerances $(T \neq M)$
- For multivariate data
- For gauge measurement errors problem
- For imprecise/fuzzy data
- For auto-corrected data
- Other applications (e.g. supplier selection, acceptance sampling plans, MPPAC, …)

Thanks for your listening!

Q & A

Available Sampling Plans by Variables for Controlling the Process Fraction Defective

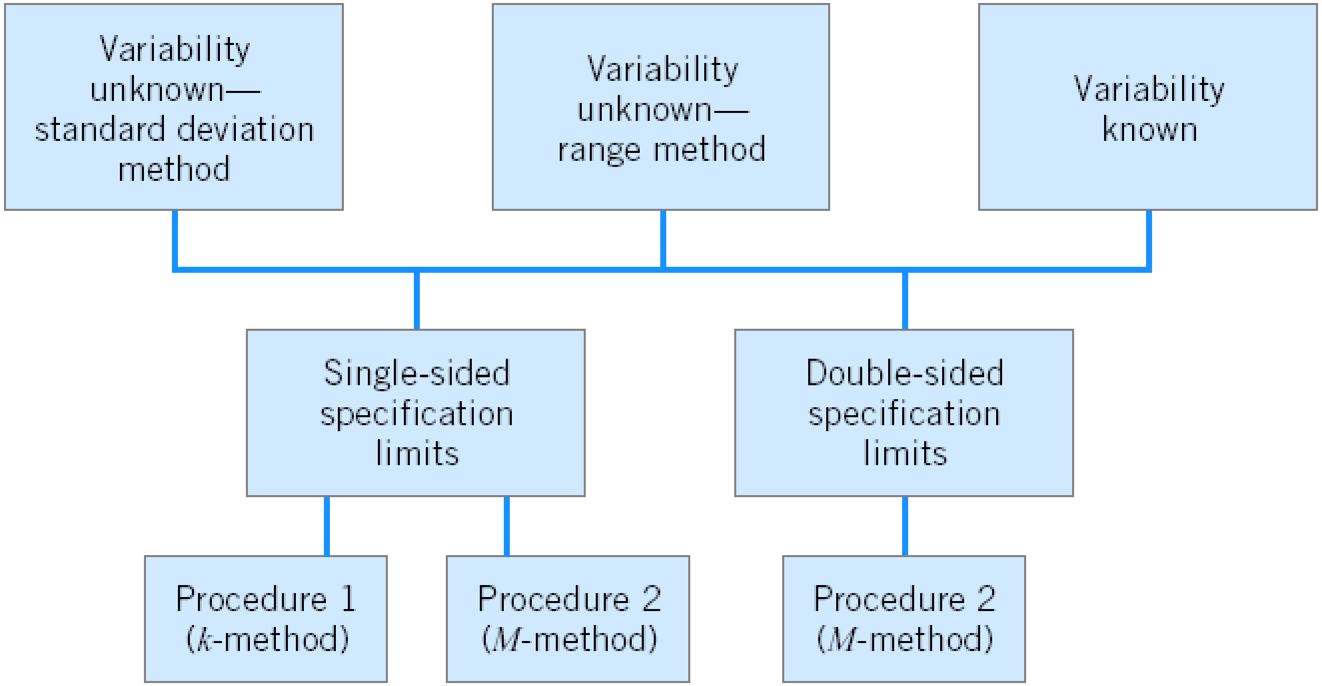


Figure 15-5 Organization of MIL STD 414.

Procedure 1 (*k* method)

Suppose that the standard deviation σ is known. Under this condition, we may wish to sample from the lot to determine whether or not the value of the mean is such that the fraction defective p is acceptable. As described next, we may organize the calculations in the variables sampling plan in two ways.

Procedure 1. Take a random sample of n items from the lot and compute the statistic

$$Z_{\mathrm{LSL}} = \frac{\bar{x} - \mathrm{LSL}}{\sigma} \tag{15-1}$$

Note that Z_{LSL} in equation 15-1 simply expresses the distance between the sample average $\bar{x}$ and the lower specification limit in standard deviation units. The larger is the value of Z_{LSL}, the farther the sample average $\bar{x}$ is from the lower specification limit, and consequently, the smaller is the lot fraction defective p. If there is a critical value of p of interest that should not be exceeded with stated probability, we can translate this value of p into a *critical distance*—say, k—for Z_{LSL}. Thus, if $Z_{\mathrm{LSL}} \geq k$, we would accept the lot because the sample data imply that the lot mean is sufficiently far above the LSL to ensure that the lot fraction nonconforming is satisfactory. However, if $Z_{\mathrm{LSL}} < k$, the mean is too close to the LSL, and the lot should be rejected.

Procedure 2 (*M* method)

Procedure 2. Take a random sample of n items from the lot and compute Z_{LSL} using equation 15-1. Use Z_{LSL} to estimate the fraction defective of the lot or process as the area under the standard normal curve below Z_{LSL}. (Actually, using $Q_{LSL} = Z_{LSL}\sqrt{n/(n-1)}$ as a standard normal variable is slightly better, because it gives a better estimate of p.) Let $\hat{p}$ be the estimate of p so obtained. If the estimate $\hat{p}$ exceeds a specified maximum value M, reject the lot; otherwise, accept it.

The two procedures can be designed so that they give equivalent results. When there is only a single specification limit (LSL or USL), either procedure may be used. Obviously, in the case of an upper specification limit, we would compute

$$Z_{USL} = \frac{USL - \bar{x}}{\sigma} \qquad (15\text{-}2)$$

instead of using equation 15-1. When there are both lower and upper specifications, the M method, Procedure 2, should be used.