應用機率模型作業 1 解答

1.

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k\ge n}^{\infty} A_k,\ \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k\ge n}^{\infty} A_k$$

$$\text{Q1: If } \sum_{n=1}^{\infty} P(A_n) < \infty,\ \text{then } P\left(\limsup_{n\to\infty} A_n\right) = 0$$

$$\text{Q2: If } \sum_{n=1}^{\infty} P(A_n) = \infty \text{ and } A_1 \coprod A_2 \coprod \cdots,\ \text{then } P\left(\limsup_{n\to\infty} A_n\right) = 1$$

Sol:

(1)

$$\because \sum_{n=1}^{\infty} P(A_n) < \infty$$

$$\therefore \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0$$

$$\text{Let } B_n = \bigcup_{k\ge n}^{\infty} A_k,$$

$$\because B_1 \supseteq B_2 \supseteq \cdots$$

$$\therefore P\left(\lim_{n\to\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$

$$\therefore 0 \le P\left(\limsup_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}^{\infty} A_k\right) = P\left(\lim_{n\to\infty}\bigcup_{k\ge n}^{\infty} A_k\right) = P\left(\lim_{n\to\infty} B_n\right)$$

$$= \lim_{n\to\infty} P(B_n) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0$$

$$\text{Hence, } P\left(\limsup_{n\to\infty} A_n\right) = 0$$

(2)

Since $\{A_n, n \ge 1\}$ are independent, $\{A_n^c, n \ge 1\}$ are also independent

$$\because \sum_{n=1}^{\infty} P(A_n) = \infty$$

$$\therefore \sum_{k=n}^{\infty} P(A_k) = \infty$$

$$\therefore 1 \ge P\left(\limsup_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{k\ge n}^{\infty} A_k\right) = P\left(\lim_{n\to\infty}\bigcup_{k\ge n}^{\infty} A_k\right) = \lim_{n\to\infty} P\left(\bigcup_{k\ge n}^{\infty} A_k\right)$$

$$= \lim_{n\to\infty}\left[1 - P\left(\bigcup_{k\ge n}^{\infty} A_k\right)^c\right] = \lim_{n\to\infty}\left[1 - P\left(\bigcap_{k\ge n}^{\infty} A_k^c\right)\right] = \lim_{n\to\infty}\left[1 - \prod_{k=n}^{\infty} P\left(A_k^c\right)\right]$$

$$= \lim_{n\to\infty}\left[1 - \prod_{k=n}^{\infty}\left(1 - P(A_k)\right)\right]$$

By $1 - x \le e^{-x}, \forall x \ge 0$

$$\lim_{n\to\infty}\left[1 - \prod_{k=n}^{\infty}\left(1 - P(A_k)\right)\right] \ge \lim_{n\to\infty}\left[1 - \exp\left(-\sum_{k=n}^{\infty} P(A_k)\right)\right] = \lim_{n\to\infty}[1-0] = 1$$

Hence, $1 \ge P\left(\limsup_{n\to\infty} A_n\right) \ge 1 \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 1$

2.

Let $\eta_X(t) = \log\left(M_X(t)\right)$, where $M_X(t)$ is the m.g.f. of random variable X

Proof :

(1) $\eta'_X(0) = E(X)$

(2) $\eta''_X(0) = Var(X)$

Sol:

(1)

$\because M_X(t)$ is the m.g.f. of random variable X

$\therefore M_X^{(k)}(0) = E\left(X^k\right), \forall k = 0,1,2,\ldots$

$$\eta'_X(0) = \eta'_X(t)\Big|_{t=0} = \frac{d}{dt}\log\left(M_X(t)\right)\Big|_{t=0} = \frac{M_X'(t)}{M_X(t)}\Bigg|_{t=0} = \frac{E(X)}{1} = E(X)$$

(2)

$$\eta''_X(0) = \eta''_X(t)\Big|_{t=0} = \frac{d^2}{dt^2}\log\left(M_X(t)\right)\Big|_{t=0} = \frac{M_X''(t)M_X(t) - \left(M_X'(t)\right)^2}{\left(M_X(t)\right)^2}\Bigg|_{t=0}$$

$$= \frac{E\left(X^2\right) - \left(E(X)\right)^2}{1^2} = Var(X)$$

3.

Suppose that a random variable X has a continuous distribution for which the cumulative distribution function is $F_X(x)$. Then the random variable Y defined as $Y = F_X(X)$ has a uniform distribution.

Proof :

Define $F_X^{-1}(y) = \inf\left\{x \mid F_X(x) \ge y\right\}$

$F_Y(y) = P(Y \le y) = P\left(F_X(X) \le y\right) = P\left(X \le F_X^{-1}(y)\right) = F_X\left(F_X^{-1}(y)\right) = y$

$\therefore Y = F_X(X) \sim Unif(0,1)$