應用機率模型作業 3 解答

Chapter2: #73, Chapter3: #24, #63

1.

(a) & (b).

$\because N_i \sim binomial(n, p_i)$

$\therefore E(N_i) = np_i, \quad Var(N_i) = np_i(1-p_i).$

(c).

$\because (N_i, N_j) \sim multinomial(n, p_i, p_j)$

$\therefore (N_i \mid N_j = n_j) \sim binomial\left(n - n_j, \frac{p_i}{1-p_j}\right)$

Hence,

$$Cov(N_i, N_j) = Cov\left[E(N_i \mid N_j), N_j\right] = Cov\left[(n - N_j)\left(\frac{p_i}{1-p_j}\right), N_j\right] = -\left(\frac{p_i}{1-p_j}\right)Var(N_j)$$

$$= -\left(\frac{p_i}{1-p_j}\right)np_j(1-p_j) = -np_i p_j.$$

(d).

Let $I_i = \begin{cases} 1, & \text{if the } i^{th} \text{ outcome do not occur.} \\ 0, & \text{otherwise.} \end{cases}$

$$\therefore E\left(\sum_{i=1}^{r} I_i\right) = \sum_{i=1}^{r} E(I_i) = \sum_{i=1}^{r} P(N_i = 0) = \sum_{i=1}^{r} (1-p_i)^n.$$

2.

(a).

Let N_{HT} denote the number of flips until at least one head and one tail have been flipped.

N_1 denote the number of flips needed "after the first flip".

$X = \begin{cases} 1, & \text{if the first outcome is head.} \\ 0, & \text{otherwise.} \end{cases}$

Since $(N_1 \mid X=0) \sim geometric(p)$, and $(N_1 \mid X=1) \sim geometric(1-p)$.

$$\therefore E(N_{HT}) = 1 + E(N_1) = 1 + EE(N_1 \mid X) = 1 + E(N_1 \mid X=0) * (1-p) + E(N_1 \mid X=1) * p$$
$$= 1 + \frac{1-p}{p} + \frac{p}{1-p}.$$

(b) & (c).

Let N_H denote the number of flips that land on heads.
N_T denote the number of flips that land on tails.

$$\because E(N_H \mid X=0) = 1, \text{ and } (N_H \mid X=1) = (N_1 \mid X=1) \sim geometric(1-p).$$

$$\therefore E(N_H) = EE(N_H \mid X) = E(N_H \mid X=0) * (1-p) + E(N_H \mid X=1) * p$$
$$= 1 - p + \frac{p}{1-p}.$$

$$E(N_T) = E(N - N_H) = \left[1 + \frac{1-p}{p} + \frac{p}{1-p}\right] - \left[1 - p + \frac{p}{1-p}\right] = \frac{1-p}{p} + p.$$

(d).

Let N_{2H1T} denote the number of flips until at least two head and one tail have been flipped.

$$\therefore E(N_{2H1T}) = EE(N_{2H1T} \mid X) = E(N_{2H1T} \mid X=0) * (1-p) + E(N_{2H1T} \mid X=1) * p$$
$$= (1 + E(\text{until two head have been flipped})) * (1-p) + (1 + E(N_{HT})) * p$$
$$= \left(1 + \frac{2}{p}\right) * (1-p) + \left(2 + \frac{1-p}{p} + \frac{p}{1-p}\right) * p.$$

3.

(a) & (b)

Let $I_i = \begin{cases} 1, & \text{if only one type } i \text{ in the final set.} \\ 0, & \text{otherwise.} \end{cases}$

$$E(I_i) = P(I_i = 1) = EP(I_i = 1 | T) = \sum_{j=0}^{n-1} P(I_i = 1 | T = j) * P(T = j) = \frac{1}{n} \sum_{j=0}^{n-1} P(I_i = 1 | T = j)$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{n-j}.$$

The final equality follows because given that there are still $n-j-1$ uncollected types when the first type i is obtained, the probability starting at that point that it will be the last of the set of $n-j$ types consisting of type i along with the $n-j-1$ yet uncollected types to be obtained is, by symmetry, $1/n-j$. Hence,

$$E\left(\sum_{i=1}^{n} I_i\right) = nE(I_i) = \sum_{j=1}^{n} \frac{1}{j}.$$