應用機率模型作業 5 解答

Chapter5: #10, #12, #43

1.

(a) .

$E\left(MX\,|M=X\right)=E\left(M^2\,|M=X\right)$

Since we know that $\left(M\,|M=X\right)$ has the same distribution with $M\sim\exp\left(\lambda+\mu\right)$

$\therefore E\left(M^2\,|M=X\right)=E\left(M^2\right)=\dfrac{2}{\left(\mu+\lambda\right)^2}.$

(b) .

Given that $M=Y\ \left(i.e.\ Y<X\right)$, by the lack of memory property

X can be denote as $M+X'$ where X' has the same distribution with X,

and is independent with M.

$$\therefore E\left(MX\,|M=Y\right)=E\left(M\left(M+X'\right)|M=Y\right)=E\left(M^2\,|M=Y\right)+E\left(MX'\,|M=Y\right)$$

$$=E\left(M^2\right)+E\left(M\right)E\left(X'\right)=\frac{2}{\left(\lambda+\mu\right)^2}+\frac{1}{\lambda\left(\lambda+\mu\right)}.$$

(c).

$Cov\left(X,M\right)=E\left(XM\right)-E\left(X\right)E\left(M\right)$

where,

$$E\left(XM\right)=EE\left(XM\,|M\right)=E\left(XM\,|M=X\right)P\left(M=X\right)+E\left(XM\,|M=Y\right)P\left(M=Y\right)$$

$$=\frac{2}{\left(\lambda+\mu\right)^2}\frac{\lambda}{\lambda+\mu}+\left(\frac{2}{\left(\lambda+\mu\right)^2}+\frac{1}{\lambda\left(\lambda+\mu\right)}\right)\frac{\mu}{\lambda+\mu}.$$

hence,

$$Cov\left(X,M\right)=\frac{2}{\left(\lambda+\mu\right)^2}\frac{\lambda}{\lambda+\mu}+\left(\frac{2}{\left(\lambda+\mu\right)^2}+\frac{1}{\lambda\left(\lambda+\mu\right)}\right)\frac{\mu}{\lambda+\mu}-\frac{1}{\lambda\left(\lambda+\mu\right)}.$$

2.

(a).

By the lack of memory property,

$$\begin{aligned}P(X_1 < X_2 < X_3) &= P\left(X_2 < X_3 \middle| X_1 = \min(X_1, X_2, X_3)\right) P\left(X_1 = \min(X_1, X_2, X_3)\right) \\ &= P(X_2 < X_3) P\left(X_1 = \min(X_1, X_2, X_3)\right) \\ &= \frac{\lambda_2}{\lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}.\end{aligned}$$

(b).

$$\begin{aligned}P\left(X_1 < X_2 \middle| X_3 = \max(X_1, X_2, X_3)\right) &= \frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)} \\ &= \frac{\frac{\lambda_2}{\lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}}{\frac{\lambda_2}{\lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{\lambda_1}{\lambda_1 + \lambda_3} \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}} \\ &= \frac{\frac{1}{\lambda_2 + \lambda_3}}{\frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_3}}.\end{aligned}$$

(c).

$E\left(\max X_i \middle| X_1 < X_2 < X_3\right) = E\left(X_3 \middle| X_1 < X_2 < X_3\right)$

by the lack of memory property, given on $X_1 < X_2 < X_3$,

$$X_3 \equiv X_2 + X_3'$$

, where $X_3' \sim \exp(\lambda_3)$ is independent with X_1, X_2, X_3.

$$\begin{aligned}\therefore E\left(\max X_i \middle| X_1 < X_2 < X_3\right) &= E(X_3') + E\left(X_2 \middle| X_1 < X_2 < X_3\right) \\ &= E(X_3') + E\left(X_2 \middle| M_1 = X_1,\ X_2 < X_3\right)\end{aligned}$$

,where M_1 denote $\min(X_1, X_2, X_3)$.

also by the lack of memory property again, given on $M_1 = X_1,\ X_2 < X_3$,

$$\begin{aligned}E\left(X_2 \middle| M_1 = X_1,\ X_2 < X_3\right) &= E\left(X_1 + X_2' \middle| M_1 = X_1,\ X_2' < X_3\right) \\ &= E\left(X_1 \middle| M_1 = X_1\right) + E\left(X_2' \middle| X_2' < X_3\right)\end{aligned}$$

, where $X_2' \sim \exp(\lambda_2)$ is independent with M_1, X_1, X_3.

Hence,

$$E\left(\max X_i \middle| X_1 < X_2 < X_3\right) = E\left(X_3'\right) + E\left(X_1 \middle| M_1 = X_1\right) + E\left(X_2' \middle| X_2' < X_3\right)$$

$$= E\left(X_3'\right) + E\left(\min\left(X_1, X_2, X_3\right)\right) + E\left(\min\left(X_2, X_3\right)\right)$$

$$= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3}.$$

(d).

$$E\left(\max X_i\right) = \sum_{i \neq j \neq k} E\left(\max\left(X_1, X_2, X_3\right) \middle| X_i < X_j < X_k\right) P\left(X_i < X_j < X_k\right)$$

$$= \sum_{i \neq j \neq k} \left(\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k}\right)\left(\frac{\lambda_j}{\lambda_j + \lambda_k} \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3}\right).$$

3.

Let T denote the time until the next arrival, S_i denote the service time at server i, $i = 1, 2$ since the exponential random variable has lack of memory property, so

$$P\left(T > S_1 + S_2\right) = P\left(T > S_1 + S_2 \middle| T > S_1\right) P\left(T > S_1\right) = P\left(T > S_2\right) P\left(T > S_1\right) = \frac{\mu_2}{\lambda + \mu_2} \frac{\mu_1}{\lambda + \mu_1}.$$