應用機率模型作業 7 解答

Chapter5: #33, #36, #56

1.

(a) .

X, Y be exponential random variables with rate λ and μ

by the lack of memory property, given that $X > Y$, $X - Y$ is still an exponential variable with rate λ, and is independent of Y.

$$\begin{aligned}\therefore P(\min(X,Y) > s, X-Y > t \mid X > Y) &= P(Y > s, X-Y > t \mid X > Y) \\ &= P(Y > s \mid X > Y) P(X-Y > t \mid X > Y) \\ &= P(\min(X,Y) > s \mid X > Y) P(X-Y > t \mid X > Y).\end{aligned}$$

Hence, given that $X > Y$, $\min(X,Y)$ and $X - Y$ are independent.

(b) .

$$\begin{aligned}E(\min(X,Y) \mid X > Y + c) &= E(\min(X,Y) \mid X > Y, X - Y > c) \\ &= E(\min(X,Y) \mid X > Y) \\ &= E(\min(X,Y)).\end{aligned}$$

(c).

By (a), we can also know that given $X < Y$, $\min(X,Y)$ and $X - Y$ are independent.

and by the previous excerise we have known that

$$P(\min(X,Y) > s \mid X > Y) = P(\min(X,Y) > s \mid X < Y) = P(\min(X,Y) > s)$$

$$\begin{aligned}&\therefore P(\min(X,Y) > s, X - Y > t) \\ &= P(\min(X,Y) > s, X-Y > t \mid X > Y) P(X > Y) + P(\min(X,Y) > s, X-Y > t \mid X < Y) P(X < Y) \\ &= P(\min(X,Y) > s \mid X > Y) P(X-Y > t \mid X > Y) P(X > Y) \\ &\quad + P(\min(X,Y) > s \mid X < Y) P(X - Y > t \mid X < Y) P(X < Y) \\ &= P(\min(X,Y) > s)\{P(X-Y > t \mid X > Y) P(X > Y) + P(X - Y > t \mid X < Y) P(X < Y)\} \\ &= P(\min(X,Y) > s) P(X - Y > t).\end{aligned}$$

Hence, $\min(X,Y)$ and $X - Y$ are independent.

2.

(a).

$$E\left(S\left(t\right)\right)=E\left(s\prod_{i=1}^{N(t)}X_i\right)=sEE\left(\prod_{i=1}^{N(t)}X_i \mid N\left(t\right)\right)=sE\left(\frac{1}{\mu}\right)^{N(t)}=s\exp\left(\lambda t\left(\frac{1}{\mu}-1\right)\right).$$

(b).

$$E\left(S^2\left(t\right)\right)=E\left(s^2\prod_{i=1}^{N(t)}X_i^2\right)=s^2EE\left(\prod_{i=1}^{N(t)}X_i^2 \mid N\left(t\right)\right)=s^2E\left(\frac{2}{\mu^2}\right)^{N(t)}=s^2\exp\left(\lambda t\left(\frac{2}{\mu^2}-1\right)\right)$$

3.

(a).

$N\left(n\right)\sim binomial\left(n,p\right).$

(b).

$T_1\sim Geo\left(p\right).$

(c).

$T_r\sim NB\left(r,p\right).$

(d).

Let $1\le d_1<d_2<\cdots<d_r\le n$

$$P\left(\text{events at } d_i, i=1,\ldots r \mid N\left(n\right)=r\right)=\frac{P\left(\text{events at } d_i, i=1,\ldots,r, N\left(n\right)=r\right)}{P\left(N\left(n\right)=r\right)}$$

$$=\frac{p^r\left(1-p\right)^{n-r}}{\binom{n}{r}p^r\left(1-p\right)^{n-r}}$$

$$=\frac{1}{\binom{n}{r}}.$$