應用機率模型作業 9 解答

Chapter4: #12, #13, #15

1.

若假設狀態空間為$\{0,1,2\}$, 其轉移矩陣為 $P=\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

依題意, 令轉移過程中皆不曾到達狀態 2

則 $Q=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \Rightarrow Q^2=\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{bmatrix}$

但 $P\left(X_2=0 \mid X_1 \neq 2, X_2 \neq 2,\ X_0=0\right)=\dfrac{P\left(X_2=0, X_1 \neq 2, X_0=0\right)}{P\left(X_1 \neq 2, X_2 \neq 2,\ X_0=0\right)}$

$$=\frac{P\left(X_2=0, X_1 \neq 2 \mid X_0=0\right)}{P\left(X_1 \neq 2, X_2 \neq 2 \mid\ X_0=0\right)}$$

$$=\frac{\left(\frac{1}{4}\right)^2}{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2+\frac{1}{4}}=\frac{1}{6} \neq Q_{0,0}^2.$$

2.

Since P^r has all positive entries, $\left(i.e.\ P_{i,j}^r>0,\ \forall i, j \in S\right)$

Let P^k has all positive entries, for some $k \geq r$, $\left(i.e.\ P_{i,j}^k>0,\ \forall i, j \in S\right)$

then,

$$P^{k+1}=P^k \times P=P \times P^k$$

where, $P_{i,j}^{k+1}=\sum_{\ell \in S} P_{i,\ell} \bullet P_{\ell,j}^k>0$ $\left(\text{since } P_{\ell,j}^k>0 \text{ and } P_{i,\ell} \text{ are not all zero}, \forall \ell \in S\right)$

$\therefore\ P^{k+1}$ has all positive entries.

By induction, P^n has all positive entries, $\forall n \geq r$.

3.

對於 M state Markov chain, 假設狀態i可到達狀態 j,
則對於i到j的所有路徑可表示為 $i_1 = i,\ i_2,\ i_3,\ \ldots,\ i_n = j$
其中, i_k 表示為轉移k次後的狀態, $i_k = 1, 2, \ldots, M$.

若其中 $i_a = i_b,\ a \neq b$, 則 i_a 至 i_b 其間為不必要的路徑, 因此路徑可縮減為

$$i_1 = i,\ i_2, \ldots,\ i_a,\ i_{b+1}, \ldots,\ i_n = j$$

依此步驟, 則對於所有路徑皆可縮減至 i_k 皆不相等, $\forall k = 1, \ldots, n$
因此為一 M state Markov chain, 故 $n \leq M$, 則此路徑最多 M step.